Production Planning and Inventories Optimization: A Backward Approach in the Convex Storage Cost Case

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Abstract

As in [3], we study the deterministic optimization problem of a profit-maximizing firm which plans its sales/production schedule. The firm knows the revenue associated to a given level of sales, as well as its production and storage costs. The revenue and the production cost are assumed to be respectively concave and convex. Here, we also assume that the storage cost is convex. This allows us to relate the optimal planning problem to the study of an integro-differential backward equation, from which we obtain an explicit construction of the optimal plan.

Key words : Production planning, inventory management, integro-differential backward equations.

Introduction

We consider a firm which produces and sells a good which can be stored. The firm acts in continuous time on a finite period in order to maximize dynamically its profit. Here, the instantaneous profit of the firm is the revenue entailed by the instantaneous sales, diminished by the cost of the instantaneous production, and by the cost of storage of the current inventories. Our approach of this production planning and inventory management problem is in the same vein as the one launched in 1958 by K. J. Harrow, S. Karlin and H. Scarf [1]. Many contributions to this theory have been brought from the 50's until now, with many different approaches. However, authors generally consider firms that do not have any control on the level of the (possibly stochastic) demand driven sales. AJOUTER DES REF, Rochet, Van der Heyden....

In this paper, we work in a competitive and deterministic context. Following, L. Arvan and L. N. Moses [2], we assume that the firm controls not only its production rate but also its sales rate. It knows the revenue associated to the selling of x units of goods, the cost of producing y units, and the cost of storing S units of the good. We assume that the marginal revenue is nonincreasing and that the marginal cost of production is nondecreasing, i.e. the revenue function is concave and the production cost function is convex.

The sales/production planning problem of the profit-maximizing firm is formulated as an optimal control problem where the controls, namely the sales and production paths are integrable. In other words, the cumulative production and sales processes are assume to be absolutely continuous.

This problem has been studied in [3] with a general storage cost function. The authors proved an existence result for a relaxed problem in which the cumulative sales process is allowed to have a jump at time 0. This means that if the firm is allowed to make a partial depletion of its eventual initial inventory at time 0. They also derived the first order conditions of optimality for both problems and thus provided a qualitative description of the optimal plans. In particular, they establish the following result : the optimal inventory level must decrease until it reaches 0. It is then kept null by producing for immediate selling.

In the present article, we go further in this analysis. For this purpose, we assume that the storage cost is convex. In this context, we first see that the initial problem has a solution if and only if the relaxed one has a solution without jumps at time 0. Second, the first order conditions of [3] allows us to characterize the (unique) optimal plan for the relaxed problem. Third, using this characterization, we relate the relaxed optimization problem to the study of a class of solutions of an integro-differential backward equation, indexed by a class of admissible terminal conditions. We establish existence and uniqueness of a maximal solution of this integro-differential backward equation, for every terminal condition. We then study the behavior of the solution with respect to the terminal condition.

This allows us to provide a constructive description of the optimal plan. The optimal plan is determined by selecting the greatest terminal condition r such that a certain functional of the solution of the integro-differential backward equation, representing the inventory level at time $0+$, $S_0(r)$, remains lower than the exogenous initial inventory s_0. The difference $\alpha = s_0 - S_0(r)$ corresponds to the size of the jump of the cumulative sales process at time 0. After 0, the sales and production rates appear to be functions of the corresponding maximal solution.

We also prove that there exists an exogenous threshold $\bar{s}_0$ on the initial inventory above which the size of the jump, α, is positive. The economic interpretation is the following. If the initial inventory s_0 entails too high storage costs, i.e. s_0 is greater than $\bar{s}_0$, then, it is optimal for the firm to sell out immediately the quantity $s_0 - \bar{s}_0$, so as to reduce its initial inventory to $\bar{s}_0$. If the initial inventory in lower than $\bar{s}_0$, then the firm has no interest in selling out some stock immediately. The level $\bar{s}_0$ appears as the maximal level that it can afford to hold.

The paper is organized as follows. Section 1 provides a precise description of the model and recalls some basic results of [3]. Section 2 is devoted to the backward characterization of the optimal plan for the relaxed problem. The constructive resolution of the production planning problem is given in Section 3.

1 The Model Formulation

The firm acts in continuous time on a finite period $[0, T]$. It is endowed with an initial inventory of $s_0 \in \mathbb{R}^+$ units of the good.

A sales/production plan is represented by a couple (x, y) of functions in $L^1_+[0, T]$, the set of nonnegative elements of $L^1[0, T]$, where $x(t)$ (resp. $y(t)$) is the sales (resp. production) rate in units of the good at time t. In other words $\int_0^t x(u)du$ (resp. $\int_0^t y(u)du$) is the cumulative quantity of the good sold out (resp. produced) up to time t. We shall say that $(x, y) \in L^1_+[0, T] \times L^1_+[0, T]$ is a sales/production plan if the induced inventory $S^{(x,y)}$ satisfies

$$S^{(x,y)}(t) \triangleq s_0 + \int_0^t y(u)du - \int_0^t x(u)du \geq 0, \quad \forall t \in [0, T].$$

This means that the company must never be out of stock. We denote by $\mathcal{A}$ the set of all sales/production plans :

$$\mathcal{A} \triangleq \{(x, y) \in L^1_+[0, T] \times L^1_+[0, T] \ \mid \ S^{(x,y)}(t) \geq 0, \forall t \in [0, T]\}.$$

When selling out at the rate $x(t)$ at time t, the firm has a revenue rate of $\pi(x(t))$. The cost of producing at the rate $y(t)$ at time t is $c(y(t))$. Both π and c are continuous, nondecreasing functions on $\mathbb{R}^+$. They satisfy $\pi(0) = 0$, $c(0) = 0$ and $\pi(x) > 0$, $c(x) > 0$, for all positive x. The function π (resp. c) is assumed to be concave (resp. convex).

The cost of storing an amount $S(t)$ of goods at time t is denoted by $s(S(t))$. The function s is assumed to be continuous, nondecreasing on $\mathbb{R}^+$ and to satisfy $s(0) = 0$ and $s(S) > 0$, for all positive S.

Given the discount rate $\lambda > 0$, the profit over time induced by $(x, y) \in \mathcal{A}$ is defined by

$$J(x,y) \triangleq \int_0^T e^{-\lambda t}[\pi(x(t)) - c(y(t)) - s(S^{(x,y)}(t))]dt.$$

Observe that by concavity of π and Jensen's inequality

$$\int_0^T e^{-\lambda t}\pi(x(t))dt < \infty\ .$$

The functions c and s being nonnegative, it follows that J is well defined as a map from $\mathcal{A}$ into $\mathbb{R} \cup \{-\infty\}$.

The profit-maximizing company plans its sales/production schedule by solving the following optimization problem

$$\sup_{(x,y)\in\mathcal{A}} J(x,y). \tag{1}$$

It turns out that the function J may fail to have a maximum on $\mathcal{A}$. This is typically the case when s_0 is to high (see section 3). Nevertheless, it was proved in [3] that existence holds for some relaxed problem that we now describe.

The sales rate is no longer described by an integrable function, but by a nonnegative finite Borel measure on $[0, T]$ which has its singular part positively proportional to the Dirac measure at 0. In this framework, for a sales path equal to $\alpha\delta_0 + x$, where $x \in L^1_+[0, T]$ represents the absolutely continuous part of the considered Borel measure, the cumulative sales process is given by

$$X(0) = 0 \text{ and } X(t) = \alpha + \int_0^t x(u)du\ ,\ \forall t \in (0, T]\ .$$

This means that the firm is allowed to sell out, at time 0, a share α of its initial inventory.

The production path is still assumed to be integrable. A sales/production plan is now a triplet (α, x, y) in $\mathbb{R}^+ \times L^1_+[0, T] \times L^1_+[0, T]$ which satisfies the inventory constraint

$$S^{(\alpha,x,y)}(t) = s_0 + \int_0^t y(u)du - \alpha - \int_0^t x(u)du \geq 0, \ \ \forall t \in (0, T].$$

For $t = 0$, we set $S^{(\alpha,x,y)}(0) = s_0$. The inventory level $S^{(\alpha,x,y)}$ can have a downward jump ($\alpha \geq 0$) at 0+.

We denote by $\mathcal{B}$ the set of relaxed sales/production plans :

$$\mathcal{B} \triangleq \{(\alpha, x, y) \in \mathbb{R}^+ \times L^1_+[0, T] \times L^1_+[0, T]\ \mid\ S^{(\alpha,x,y)}(t) \geq 0, \forall t \in (0, T]\}.$$

The relaxed profit is defined on $\mathcal{B}$ by

$$\mathcal{F}(\alpha, x, y) = \alpha\dot{\pi}(\infty) + \int_0^T e^{-\lambda t}[\pi(x(t)) - c(y(t)) - s(S^{(\alpha,x,y)}(t))]dt$$

where, we have set

$$\dot{\pi}(\infty) \triangleq \lim_{x\to\infty} \pi(x)/x$$

which is well defined in $\mathbb{R}^+$, by concavity and nonnegativity of π. This is the price at which the firm can sell at an infinite rate. It is also the lowest price accessible for the company. However, since holding inventories has a cost, the firm may take advantage of an immediate depletion, even at this price. Observe that, when π is differentiable, we have

$$\lim_{x\to\infty} \frac{\pi(x)}{x} = \lim_{x\to\infty} \dot{\pi}(x).$$

The relaxed optimization problem is

$$\sup_{(\alpha,x,y)\in\mathcal{B}} \mathcal{F}(\alpha, x, y). \tag{2}$$

It was proved in [3] that

Theorem 1 *$\mathcal{F}$ has a maximum in $\mathcal{B}$ and* $\max_{(\alpha,x,y)\in\mathcal{B}} \mathcal{F}(\alpha, x, y) = \sup_{(x,y)\in\mathcal{A}} J(x, y)$.

Notice that in the context of [3], where the storage cost is not assumed to be convex, uniqueness is not guaranteed in Problem (2). In this article, we shall go further in the analysis of the set of solutions of both problems. For this purpose, we require some regularity conditions on π, c and s and we assume that s is convex. In particular, this will allow us to assert that Problem (2) has a unique solution and thus that Problem 1 has a (unique) solution if and only if the solution of Problem (2) is in $\{0\} \times \mathcal{A}$.

To be more precise, we shall work under the following conditions.

Standing assumption (H)**.**
(i) Argmax $(\pi - c) = \{a\}$ *for some* $a > 0$.
(ii) π *is differentiable on* $(0, \infty)$ *and* $\dot{\pi}$ *is continuous and one to one on* $[a, \infty)$.
(iii) c *is differentiable on* $\mathbb{R}^+$ *and* $\dot{c}$ *is continuous and one to one on* $[0, a]$.
(iv) s *is continuously differentiable on* $\mathbb{R}^+$.
(v) s *is convex.*

Assumptions (i) to (iv) were already required in [3] in order to obtain a precise characterization of the set of plans which satisfy the first order conditions of optimality in Problem (2). The existence result has been obtained under a weaker assumption : the functions π, c and s are continuous, and $\pi - c$ admits a, possibly not unique, maximum.

Theorem 2 *Problem (2) has a unique solution (α, x, y). Problem (1) has a solution if and only if $\alpha = 0$. If $\alpha = 0$ then, (x, y) is the unique solution of (1).*

Proof. Existence for Problem (2) is stated in Theorem 1 and was proved in [3]. Moreover, Theorem 7 in [3] states that, if (α, x, y) is a solution then $x \geq a \geq y$ a.e. Since π and c are respectively strictly concave on $[a, \infty)$ and strictly convex on $[0, a]$, and since s is convex, uniqueness in Problem (2) is straightforward. By noticing that

$$(0, x, y) \in \mathcal{B} \Leftrightarrow (x, y) \in \mathcal{A} \text{ and } \mathcal{F}(0, \cdot)_{|\mathcal{A}} \equiv J$$

and recalling that by Theorem 1 $\max_{(\alpha,x,y)\in\mathcal{B}} \mathcal{F}(\alpha, x, y) = \sup_{(x,y)\in\mathcal{A}} J(x, y)$ the proof of Theorem 2 is easy to complete. $\square$

In light of Theorem 2, we see that planning the optimal sales/production schedule can be done by solving the relaxed problem. Indeed, if the initial problem (1) has a solution (x, y) then, $(\alpha = 0, x, y)$ is the solution of the relaxed problem (2). Since, we further exhibit situations where the solution of Problem (2) is not regular i.e. $\alpha > 0$ (see section 3), meaning that Problem (1) has no solution, we must in actual fact consider Problem (2) to solve the planning problem. We shall see that for some revenue and costs functions π, c and s and some initial inventory s_0, the firm can not act in an optimal way without getting rid at time 0 of a certain share of s_0. From now on, we focus our attention on the relaxed problem and what we call the optimal plan is the unique solution (α, x, y) of Problem (2).

2 Characterization of the Optimal Plan.

In this section we begin with recalling some characterization and thus some qualitative description of the optimal plan derived from the first order conditions obtained in [3]. We then introduce some more tractable formulation of these optimality conditions in order to reach our main goal here : to provide a constructive resolution of the planning problem.

2.1 The first order conditions

It was shown in [3] that the optimal plan must be such that there is no inventory accumulation. In particular, if the firm has no starting inventories ($s_0 = 0$) then it adopts a static strategy consisting in producing for immediate sales. Moreover, the concavity of the revenue and convexity of the cost of production urges the firm to minimize the variations of its sales and production rates, so that it must produce and sell at the same constant rate a which maximize $\pi - c$. To sum up, when $s_0 = 0$, the optimal plan is $(0, a, a)$. Let us now turn to the case where s_0 is positive.

Standing assumption. *The initial inventory s_0 is positive.*

As a direct consequence of Theorem 7 in [3], we have the following characterization of the solution of Problem (2).

Proposition 3 $(\alpha, x, y) \in \mathcal{B}$ *is the solution of (2) if and only if it satisfies :*

(I) x *(resp.* y*) is nonincreasing (resp. nondecreasing) and* $x \geq a \geq y$ *on* $[0, T]$,

(II) $S^{(\alpha,x,y)}(T) = 0$ *and* $T_0 \triangleq \inf\{t \in (0, T] \mid S^{(\alpha,x,y)}(t) = 0\} > 0$,

(III) *if* $T_0 < T$ *then,*

a) $S^{(\alpha,x,y)} = 0$ *on* $[T_0, T]$,

b) $x = y = a$ *on* $(T_0, T]$,

(IV) $S^{(\alpha,x,y)}$ *is decreasing on* $(0, T_0]$,

(V) *a)* x *and* y *are both continuous on* $(0, T_0)$ *and satisfy*

$$\text{b)} \quad e^{-\lambda t}\dot{\pi}(x(t)) = \dot{\pi}(x(0+)) + \int_0^t e^{-\lambda u}\dot{s}\left(S^{(\alpha,x,y)}(u)\right) du, \ \forall t \in (0, T_0) \tag{3}$$

$$\text{c)} \quad y(t) = g(x(t)), \ \forall t \in (0, T_0) \tag{4}$$

$$\textit{where } g(r) \triangleq \begin{cases} \dot{c}^{-1}(\dot{\pi}(r)) & \textit{if } \ \dot{\pi}(r) > \dot{c}(0) \\ 0 & \textit{elsewhere} \end{cases}, \ \forall\, r \in [a, \infty). \tag{5}$$

(VI) *If* $T_0 < T$ *then,* $x(T_0-) = a$.

(VII) *If* $\alpha > 0$ *then,* $x(0+) = \infty$.

Proof. By Theorem 7 in [3], items (I) to (VII) are equivalent to :

$$\limsup_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} \left\{ \frac{\mathcal{F}((\alpha, x, y) + \varepsilon(\beta, h, k)) - \mathcal{F}(\alpha, x, y)}{\varepsilon} \right\} \leq 0,$$

for all $(\beta, h, k) \in \mathbb{R} \times L^1[0, T] \times L^1[0, T]$ such that $(\alpha, x, y) + \varepsilon_0(\beta, h, k) \in \mathcal{B}$ for some $\varepsilon_0 > 0$. Then, Proposition 3 holds by concavity of $\mathcal{F}$ and uniqueness in Problem (2). □

Proposition 3 furnishes some qualitative properties of the optimal plan which were obtained in [3] without assuming that the cost of storage is convex. The main property is that the inventory level is nonincreasing and must be null at the end of the period. This phenomenon arises directly from the concavity of the revenu and the convexity of the cost of production and the fact that holding inventories has a cost.

EXPLIQUER ???

Let us now put in light the qualitative description provided by Proposition 3. The optimal way to deplete the initial inventory is in two phases. This leads to a three phases sales/production plan. The first phase is devoted to the selling activity. The firm begins with possibly depleting a share $\alpha \geq 0$ of its initial inventory at time 0, selling at an infinite rate. The sales rate is then nonincreasing or equivalently the marginal revenue $t \longrightarrow \dot{\pi}(x(t))$ is nondecreasing. If the marginal revenue overtakes the lowest marginal cost of production $\dot{c}(0)$ then, the production activity actually starts (see (4) and (5)). During this second phase, the sales rate and the production rate are such that the marginal revenue and the marginal production cost remain equal : $\dot{c}(y(t)) = \dot{\pi}(x(t))$. The production rate is nondecreasing. During this destocking stage the sales rate is greater than a and the production rate lower. If inventories are all cleared before the end of the period ($T_0 < T$) then, the third phase starts : production and sales are at the same constant rate, a, maximizing the instantaneous profit.

Remark that by (ii), even if the firm is allowed to get rid of its whole initial inventory at time 0, it does not. This is mainly due to the fact that α is sold out time 0 at an infinite rate and hence at the lowest price $\dot{\pi}(\infty)$.

2.2 The Backward Characterization

We now turn to our first step on the way to the explicit determination of the optimal plan (α, x, y), that is providing a backward procedure to find (α, x, y). We shall extract from Proposition 3 some sufficient conditions of optimality which are expressed by mean of some integro-differential backward equation that must be satisfied by the sales rate x. We first obtain these conditions by arguing in the necessary way. We then check that they are sufficient.

In the sequel we will use the following properties of the function g defined in (5).

Remark 4 *Under assumption (H) and by construction the function g is continuous, non-increasing on $[a, \infty)$. Moreover, it satisfies $g(a) = \dot{c}^{-1}(\dot{\pi}(a)) = a$ and takes values in $[0, a]$.*

First of all, remark that by Proposition 3 the optimal production rate y is an exogenous function of the optimal sales rate x. More precisely, if $T_0 = T$ then, by (4) we have $y(t) = g(x(t))$ for any $t \in (0, T)$. If $T_0 < T$ then, by (4) again we have $y(t) = g(x(t))$ for any $t \in (0, T_0)$ and by (iiib) and since $a = g(a)$, we also have $y(t) = a = g(a) = g(x(t))$ for any $t \in (T_0, T]$. In both cases we have :

$$y = g(x) \text{ a.e. on } [0, T]. \tag{6}$$

In light of this result, we now seek for some conditions on some sales policy $(\tilde{\alpha}, \tilde{x})$ for the plan $(\tilde{\alpha}, \tilde{x}, g(\tilde{x}))$ to be the optimal one.

Observe that, by (IIIa) we have $S^{(\alpha,x,y)}(T_0) = 0$ and therefore, for any $t \in [0, T]$,

$$\begin{aligned} S^{(\alpha,x,y)}(t) &= S^{(\alpha,x,g(x))}(t) \\ &= S^{(\alpha,x,g(x))}(t) - S^{(\alpha,x,g(x))}(T_0) \\ &= s_0 - \alpha + \int_0^t [g(x(u)) - x(u)]du - \left(s_0 - \alpha + \int_0^{T_0} [g(x(u)) - x(u)]du\right) \\ &= \int_t^{T_0} [x(u) - g(x(u))]du \,. \end{aligned}$$

Using this backward formulation of the inventory level, it is easy to see that (α, x) satisfies the backward system deduced from (3) and from the equality $S^{(\alpha,x,g(x))}(T_0) = 0$

$$\begin{cases} e^{-\lambda t}\dot{\pi}(x(t)) &= e^{-\lambda T_0}\dot{\pi}(x(T_0-)) - \int_t^{T_0} e^{-\lambda u}\dot{s}\left(\int_u^{T_0}[x(s) - g(x(s))]ds\right)du, \ \forall t \in (0, T_0) \\ \alpha &= s_0 - \int_0^{T_0}[x(u) - g(x(u))]du \end{cases}$$

Now, by using appropriated changes of variable, one can check that, in term of the translation of $x_{|(0,T_0)}$ on $(T - T_0, T)$ which is defined by $w(t) = x(t-(T-T_0))$, for any $t \in (T-T_0, T]$, this system reads :

$$\begin{cases} e^{-\lambda t}\dot{\pi}(w(t)) &= e^{-\lambda T}\dot{\pi}(w(T-)) - \int_t^{T} e^{-\lambda u}\dot{s}\left(\int_u^{T}[w(s) - g(w(s))]ds\right)du, \ \forall t \in (T - T_0, T) \\ \alpha &= s_0 - \int_{T-T_0}^{T}[w(u) - g(w(u))]du \end{cases}$$

We finally see that the translation w of $x_{|(0,T_0)}$ on $(T - T_0, T)$ can be found in the set of solutions of the above integro-differential backward equation indexed by the set $[a, \infty)$ of possible terminal conditions (by (I) x takes values in $[a, \infty)$). We shall see that w is characterized, among these solutions, by some boundary conditions derived from (IIIa), (VI) and (VII), which moreover furnish α. This is Corollary 6 below.

For sake of readability we introduce a notation for the difference between the optimal sales rate and the optimal production rate. We shall put :

$$\delta(r) \triangleq r - g(r) \ \ , \ \ \forall\, r \in [a, \infty).$$

We have $x - y = \delta(x)$ a.e. on $[0, T]$. The following remark embodies some useful properties of δ.

Remark 5 *By Remark 4, δ is continuous and increasing on $[a, \infty)$, satisfies $\delta(a) = 0$ and hence is positive on (a, ∞). Moreover $\lim_{r\to\infty} \delta(r) = \infty$.*

We can now state our backward procedure to find the optimal plan.

Corollary 6 *Consider some function w and some couple $(\tau_0, r) \in [0, T) \times [a, \infty)$ such that :*

(i) $w(T) = r$, *w is continuous and decreasing on $(\tau_0, T]$,*

(ii)

$$\text{if } r > a \text{ then } \tau_0 = 0, \tag{7}$$

$$e^{-\lambda t}\dot{\pi}(w(t)) = e^{-\lambda T}\dot{\pi}(r) - \int_t^T e^{-\lambda u}\dot{s}\left(\int_u^T \delta(w(s))ds\right)du,\ \forall t \in (\tau_0, T], \tag{8}$$

$$\int_{\tau_0}^T \delta(w(u))du \leq s_0, \tag{9}$$

$$\text{if } \int_{\tau_0}^T \delta(w(u))du < s_0 \text{ then } w(\tau_0+) = \infty. \tag{10}$$

Then,

$$\begin{pmatrix} \alpha \\ x \\ y \end{pmatrix} = \begin{pmatrix} s_0 - \int_{\tau_0}^T \delta(w(u))du \\ w(\cdot+\tau_0)\mathbf{1}_{(0,T-\tau_0]} + a\mathbf{1}_{(T-\tau_0,T]} \\ g\left(w(\cdot+\tau_0)\mathbf{1}_{(0,T-\tau_0]} + a\mathbf{1}_{(T-\tau_0,T]}\right) \end{pmatrix} \tag{11}$$

Proof. Let w and (τ_0, r) be as in the statement. We shall prove that the plan $(\tilde{\alpha}, \tilde{x}, \tilde{y})$ defined by

$$\begin{aligned} \tilde{\alpha} &\triangleq s_0 - \int_{\tau_0}^T \delta(w(u))du && (12)\\ \tilde{x} &\triangleq w(\cdot+\tau_0)\mathbf{1}_{(0,T-\tau_0]} + a\mathbf{1}_{(T-\tau_0,T]} && (13)\\ \tilde{y} &\triangleq g(\tilde{x}) && (14) \end{aligned}$$

is in $\mathcal{B}$ and satisfies conditions (I) to (VII) of Proposition 3 which characterize (α, x, y).

Since w is decreasing on $(\tau_0, T]$ and $w(T) = r \in [a, \infty)$, by (13) $\tilde{x}$ is nonincreasing and takes values in $[a, \infty)$ on $(0, T]$. By Remark 4, g is nonincreasing, taking values in $[0, a]$ on $[a, \infty)$. Therefore, by (14) $\tilde{y}$ is nondecreasing and takes values in $[0, a]$ on $(0, T]$. Besides, by (9), by definition of δ and since $\tau_0 \geq 0$ and since g takes values in $[0, a]$ we have

$$\int_{\tau_0}^T w(u)du \leq s_0 + \int_{\tau_0}^T g(w(u))du \leq s_0 + Ta.$$

Since $w \geq 0$, this implies that $w \in L^1_+[\tau_0, T]$ and then, by construction $\tilde{x} \in L^1_+[0, T]$. By (9) again, we have $\tilde{\alpha} \geq 0$. Thus $(\tilde{\alpha}, \tilde{x}, \tilde{y}) \in \mathbb{R}^+ \times L^1_+[0, T] \times L^1_+[0, T]$ and $\tilde{x}$ and $\tilde{y}$ satisfy (I).

Let us now prove that

$$S^{(\tilde{\alpha},\tilde{x},\tilde{y})} \geq 0 \text{ on } (0, T] \text{ and that (II), (III) and (IV) hold.} \tag{15}$$

We shall obtain that $\tilde{T}_0 \triangleq \inf\{t \in (0, T] \mid S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(t) = 0\} = T - \tau_0$. We begin with proving that $\tilde{x} = \tilde{y} = a$ and $S^{(\tilde{\alpha},\tilde{x},\tilde{y})} = 0$ on $(T - \tau_0, T]$. By (13) and (14) and since $g(a) = a$ we have :

$$\tilde{x} = a = g(a) = g(\tilde{x}) = \tilde{y} \text{ on } (T - \tau_0, T], \tag{16}$$

so that

$$S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(t) = S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(T-\tau_0),\ \forall t \in (T-\tau_0, T]. \tag{17}$$

By (14), (12), (13) and after a change of variable, for any $t \in (0, T-\tau_0]$ we have :

$$\begin{aligned} S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(t) &= s_0 - \tilde{\alpha} + \int_0^t [g(\tilde{x}(u)) - \tilde{x}(u)]du \\ &= \int_{\tau_0}^T \delta(w(u))du + \int_0^t [g(w(u+\tau_0)) - w(u+\tau_0)]du \\ &= \int_{\tau_0}^T \delta(w(u))du + \int_{\tau_0}^{t+\tau_0} [g(w(u)) - w(u)]du, \end{aligned}$$

that is by definition of δ,

$$S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(t) = \int_{t+\tau_0}^T \delta(w(u))du\ . \tag{18}$$

Therefore $S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(T-\tau_0) = 0$ and by (17)

$$S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(t) = 0,\ \forall t \in [T-\tau_0, T]. \tag{19}$$

In order to complete the proof of (15) it suffices to check that

$$S^{(\tilde{\alpha},\tilde{x},\tilde{y})} \text{ is decreasing on } (0, T-\tau_0]. \tag{20}$$

Indeed from (19) and since by assumption $\tau_0 \in [0,T)$ we will therefore deduce that

$$S^{(\tilde{\alpha},,\tilde{x},\tilde{y})} \geq 0 \text{ on } (0,T],\quad \tilde{T}_0 \triangleq \inf\{t \in (0,T] \mid S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(t) = 0\} = T - \tau_0 \quad \text{and} \quad \tilde{T}_0 > 0,$$

so that (II) and (IIIa) hold by (19) again, (IIIb) holds by (16) and (IV) holds by (20).

As for (20), recall that by Remark 5, δ takes positive values on (a,∞). Moreover, since by (i) w is decreasing on $(\tau_0, T]$ and satisfies $w(T) = r \geq a$, it takes values in (a,∞) on (τ_0, T). Consequently, $\delta \circ w$ is positive on (τ_0, T). It then follows from (18) that $S^{(\tilde{\alpha},\tilde{x},\tilde{y})}$ is decreasing on $(0, T-\tau_0]$. This concludes the proof of (15).

Let us now turn to the proof of (V). Item (Vc) holds by construction (see 14). Since w is continuous on $(\tau_0, T]$, by (13) $\tilde{x}$ is continuous on $(0, T-\tau_0] = (0, \tilde{T}_0]$ and so is $\tilde{y} = g(\tilde{x})$ since g is continuous on $[a,\infty)$. This proves (Va). Let us establish (Vb), i.e. let us prove that (3) is satisfied. By (13), equation (8) reads :

$$e^{-\lambda t}\dot{\pi}(\tilde{x}(t-\tau_0)) = e^{-\lambda T}\dot{\pi}(r) - \int_t^T e^{-\lambda u}\dot{s}\left(\int_u^T \delta(w(s))ds\right)du,\ \forall t \in (\tau_0, T],$$

which, after a change of variable from $(\tau_0, T]$ onto $(0, T-\tau_0] = (0, \tilde{T}_0]$ yields :

$$e^{-\lambda(t+\tau_0)}\dot{\pi}(\tilde{x}(t)) = e^{-\lambda T}\dot{\pi}(r) - \int_{t+\tau_0}^T e^{-\lambda u}\dot{s}\left(\int_u^T \delta(w(s))ds\right)du,\ \forall t \in (0, \tilde{T}_0],$$

equivalently,

$$e^{-\lambda t}\dot\pi(\tilde x(t)) = e^{-\lambda(T-\tau_0)}\dot\pi(r) - \int_{t+\tau_0}^{T} e^{-\lambda(u-\tau_0)}\dot s\left(\int_u^T \delta(w(s))ds\right)du,\ \forall t\in(0,\tilde T_0].$$

Using a change of variable in the integral and since $T-\tau_0=\tilde T_0$ we obtain :

$$e^{-\lambda t}\dot\pi(\tilde x(t)) = e^{-\lambda \tilde T_0}\dot\pi(r) - \int_{t}^{\tilde T_0} e^{-\lambda u}\dot s\left(\int_{u+\tau_0}^T \delta(w(s))ds\right)du,\ \forall t\in(0,\tilde T_0].$$

Plugging (18) in this last equation we get :

$$e^{-\lambda t}\dot\pi(\tilde x(t)) = e^{-\lambda \tilde T_0}\dot\pi(r) - \int_{t}^{\tilde T_0} e^{-\lambda u}\dot s\left(S^{(\tilde\alpha,\tilde x,\tilde y)}(u)\right)du,\ \forall t\in(0,\tilde T_0]. \tag{21}$$

Observe that since $\tilde x$ is nonincreasing and takes values in $[a,\infty)$, since $\dot\pi$ is continuous and decreasing on $[a,\infty)$ and satisfies $\lim_{x\to\infty}\dot\pi(x)=\dot\pi(\infty)$ we have :

$$\lim_{t\searrow 0}\dot\pi(\tilde x(t)) = \dot\pi(\tilde x(0+))\in[\dot\pi(\infty),\dot\pi(a)].$$

Therefore, sending t to 0 in (21) we obtain :

$$\dot\pi(\tilde x(0+)) = e^{-\lambda \tilde T_0}\dot\pi(r) - \int_{0}^{\tilde T_0} e^{-\lambda u}\dot s\left(S^{(\tilde\alpha,\tilde x,\tilde y)}(u)\right)du\ . \tag{22}$$

By computing the difference between (21) and (22), we obtain (3).

To complete the proof of Corollary 6 it remains to show that (VI) and (VII) are satisfied i.e. to check that :

$$\begin{aligned} \text{if }\ \tilde T_0 < T\ \text{ then }\ \tilde x(\tilde T_0-) &= a\,, \qquad (23)\\ \text{if }\ \tilde\alpha>0\ \text{ then }\ \tilde x(0+) &= \infty\,. \qquad (24)\end{aligned}$$

Observe that, by (i) and (13)

$$\tilde x(\tilde T_0-) = w(T-) = w(T) = r\in[a,\infty). \tag{25}$$

Therefore, since by (7) [$r>a \Rightarrow \tau_0=0$] and since [$\tilde T_0<T \Leftrightarrow \tau_0>0$], condition (23) is satisfied. As for the second condition, we have by (13) again : $\tilde x(0+)=w(\tau_0+)$ and by (12)

$$\tilde\alpha = s_0 - \int_{\tau_0}^{T}\delta(w(u))du$$

Condition (24) therefore holds since it is equivalent to condition (10). This ends the proof of Corollary 6. □

3 Constructive resolution of the production planning problem.

In the sequel, for $r \in [a, \infty)$, $BW(r)$ stand for the integro-differential equation involved in Corollary 6 :

$$BW(r) \quad : \quad e^{-\lambda t}\dot{\pi}(w(t)) = e^{-\lambda T}\dot{\pi}(r) - \int_t^T e^{-\lambda u}\dot{s}\left(\int_u^T \delta(w(s))ds\right) du\,, \;\; t \le T$$

By studying the set of solutions of the equation $BW(r)$ when r varies in $[a, \infty)$ in order to determine the optimal plan (α, x, y) by applying Corollary 6 , we achieve here our main aim : to provide a constructive resolution of the production planning problem.

In addition, we shall give necessary and sufficient conditions for the optimal plan to be *regular* ($\alpha = 0$) i.e. without depletion at time 0. Given the revenue and costs functions and the length of the planning period, the regularity of the optimal plan, depends only on the level of the initial inventory s_0. We shall obtain that there is a positive (possibly infinite) level $\bar{s}_0$, depending on π, c, s and T such that : if $s_0 \le \bar{s}_0$ (resp. $s_0 > \bar{s}_0$) then the solution is regular : $\alpha = 0$ (resp. not regular : $\alpha > 0$). We shall also exhibit some qualitative distinction between the optimal schedule obtained on a long planning period or on a short one.

In the sequel, we use the following notation : for all interval I and all set $E \subset \mathbb{R}$, we denote by $C(I; E)$ the set of functions that are continuous on I with values in E. For $E = \mathbb{R}$, we simply write $C(I)$ for $C(I; E)$. The proofs of the technical results on equation BW that are stated in this section are postponed to Appendix A and B.

We start with the existence and uniqueness of a maximal solution for $BW(r)$, given any terminal condition $r \in [a, \infty)$. From this maximal feature we deduce a necessary and sufficient condition for the solution to explode at some given time. This property may be needed when applying Corollary 6 (see 10). We also check that the solution has the continuity and monotony required in Corollary 6. For this purpose we work under the following technical

Standing Assumptions
$(H\pi)$ *The inverse of* $\dot{\pi}_{|[a,\infty)}$ *is locally Lipschtiz on* $(\dot{\pi}(\infty), \dot{\pi}(a)]$.
(Hc) *The inverse of* $\dot{c}_{|[0,a]}$ *is locally Lipschitz on* $[\dot{c}(0), \dot{c}(a)]$.

Theorem 7 *For every* $r \in [a, \infty)$, *there exists a unique couple* $(\tau(r), w) \in [-\infty, T) \times C((\tau(r), T])$, *such that* w *satisfies* $BW(r)$ *on* $(\tau(r), T]$ *:*

$$e^{-\lambda t}\dot{\pi}(w(t)) = e^{-\lambda T}\dot{\pi}(r) - \int_t^T e^{-\lambda u}\dot{s}\left(\int_u^T \delta(w(s))ds\right) du\,,\; \forall t \in (\tau(r), T],$$

and such that

$$\text{either} \ \ \tau(r) = -\infty, \quad \text{or} \ \ \tau(r) > -\infty \ \ \text{and} \ \ w(\tau(r)+) = \infty. \tag{26}$$

In addition, w is decreasing and takes values in $[a, \infty)$.

Notation

Given $r \in [a, \infty)$, if $(\tau(r), w)$ is associated to r by Theorem 7 we shall write in short w^r and say that w^r is the maximal solution of $BW(r)$.
We denote by τ the map from $[a, \infty)$ into $[-\infty, T)$ defined by Theorem 7.

In the sequel we shall use the following

Remark 8 *Let $r \in [a, \infty)$. Since w^r is continuous, decreasing and takes values in $[a, \infty)$ on $(\tau(r), T]$ and since by Remark 5 δ is continuous, increasing and nonnegative on $[a, \infty)$, we see that $\delta \circ w^r$ is continuous, decreasing and nonnegative on $(\tau(r), T]$. It is therefore positive on $(\tau(r), T)$.*

Theorem 7 is sufficient to initialized the constructive resolution of the planning problem by using Corollary 6. Indeed, one may begin with considering the solution w^a of $BW(a)$ and answer the question :

Does there exist some $\tau_0 \in [\max\{0, \tau(a)\}, T)$ such that :

$$\begin{array}{lll} \text{(a)} & \text{either} & \int_{\tau_0}^{T} \delta(w^a(u))du = s_0, \\ \text{(b)} & \text{or} & \int_{\tau_0}^{T} \delta(w^a(u))du < s_0 \ \text{ and } \ w^a(\tau_0+) = \infty \ \ ? \end{array} \tag{27}$$

Remark 9 *Assume that the answer to (27) is positive. First, by Theorem 7, w^a is continuous, decreasing on $(\tau_0, T]$ and such that $w^a(T) = a$, so that (i) of Corollary 6 holds for w^a with τ_0 and $r = a$. Second, by definition and by (27), w^a satisfies (8), (9) and (10) so that (ii) also holds. The optimal plan is then given by (11) in function of w^a.*

The situations where the answer to (27) is affirmative are described in Theorems 12 and 13 below, obtained by using monotony and continuity of the (translated) backward inventory process associated to w^a.

Remark 10 *By Remark 8, the function*

$$t \in (\tau(a), T] \longrightarrow S(t, a) \triangleq \int_t^T \delta(w^a(s))ds \in \mathbb{R}^+$$

is well-defined, decreasing and continuous on $(\tau(a), T]$. It has a limit as t goes to $\tau(a)$ which is :

$$s^a \triangleq \sup_{t \in (\tau(a), T]} \int_t^T \delta(w^a(u))du = \int_{\tau(a)}^T \delta(w^a(u))du \in \mathbb{R}^+ \cup \{\infty\}$$

Since $\tau(a) < T$, it is positive.

We claim that there are two cases to keep distinct : either $\tau(a) > 0$ and then the answer to (27) is positive, or $\tau(a) \leq 0$ and the answer may be negative. If The alternative between $\tau(a) > 0$ and $\tau(a) \leq 0$ can be economically interpreted as the one between a long planning period and a short one. Indeed, saying that $\tau(a) > 0$ (resp. $\tau(a) \leq 0$) amounts to saying that the length of the planning period T is greater (resp. lower) than the length $T_a \triangleq T - \tau(a)$ of the definition interval of the maximal solution w^a of $BW(a)$. It turns out that T_a does in actual fact not depend on T. This comes from the time homogeneity of BW (invariance by translation on the terminal time) : by using Theorem 7 and changes of variable, it is easy to prove the following

Remark 11 *Let $r \in [a, \infty)$ and T_1, $T_2 \in [0, \infty)$. If $(w_1, \tau_1(r))$ (resp. $(w_2, \tau_2(r))$) is the maximal solution of $BW(r)$ with terminal time $T = T_1$ (resp. T_2) then we have : $T_1 - \tau_1(r) = T_2 - \tau_2(r)$ and $w_1(t) = w_2(t + T_2 - T_1)$, $\int_t^{T_1} \delta(w_1^r(u))du = \int_{t+T_2-T_1}^{T_2} \delta(w_2^r(u))du$ for any $t \in (\tau_1(r), T_1]$.*

Therefore the length of the domain of the maximal solution of $BW(a)$, $T_a = T - \tau(a)$ and the level $s^a = \int_{\tau(a)}^T \delta(w^a(u))du = \int_0^{T_a} \delta(w_{T_a}^a(u))du$ depend only on π, c and s and not on the length of the planning period T. In the sequel, we shall say that the planning period is *long* (resp. *short*) if $\tau(a) > 0$ (resp. $\tau(a) \leq 0$).

3.1 The Long Planning Period Case

Theorem 12 *Assume that $\tau(a) > 0$.*

1. *If $s_0 \leq s^a = \int_{\tau(a)}^T \delta(w^a(u))du$ then, there exists a unique $\tau_0 \in [0, T)$ such that $\int_{\tau_0}^T \delta(w^a(u))du = s_0$ and the optimal plan is regular, given by*

$$\begin{pmatrix} \alpha \\ x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ w^a(\cdot + \tau_0)\mathbf{1}_{(0,T-\tau_0]} + a\mathbf{1}_{(T-\tau_0,T]} \\ g\left(w^a(\cdot + \tau_0)\mathbf{1}_{(0,T-\tau_0]}\right) + a\mathbf{1}_{(T-\tau_0,T]} \end{pmatrix}$$

2. *If $s_0 > s^a = \int_{\tau(a)}^T \delta(w^a(u))du$ then, the optimal plan is not regular, given by*

$$\begin{pmatrix} \alpha \\ x \\ y \end{pmatrix} = \begin{pmatrix} s_0 - \int_{\tau(a)}^T \delta(w^a(u))du \\ w^a(\cdot + \tau(a))\mathbf{1}_{(0,T-\tau(a)]} + a\mathbf{1}_{(T-\tau(a),T]} \\ g\left(w^a(\cdot + \tau(a))\mathbf{1}_{(0,T-\tau(a)]}\right) + a\mathbf{1}_{(T-\tau(a),T]} \end{pmatrix}$$

Proof. Recall that $s_0 > 0 = S(T, a)$. If $s_0 \leq \int_{\tau(a)}^T \delta(w^a(u))du = S(\tau(a)+, a)$ the existence of a unique $\tau_0 \in [\tau(a), T)$ such that $s_0 = \int_{\tau_0}^T \delta(w^a(u))du = S(\tau_0, a)$ is insured by Remark 10 and the Mean-Value Theorem. In addition, since $\tau(a) > 0$, we have $\tau_0 > 0$. Therefore condition (27a) holds for τ_0 and the proof of item 1 is ended by using Remark 9. Since $\tau(a) > -\infty$, by (26) we have $w^a(\tau(a)+) = \infty$. Therefore, if $s_0 > \int_{\tau(a)}^T \delta(w^a(u))du$, condition (27b) holds for $\tau_0 = \tau(a)$. Using Remark 9 again we then obtain item 2. □

From Theorem 12, we see that when the planning period is long ($\tau(a) > 0$), the optimal plan is regular if and only if the initial inventory is below the exogenous level

$$\overline{s_0} = s^a = \int_{\tau(a)}^T \delta(w^a(s))ds.$$

Notice that it may be infinite. In that case, whatever the initial inventory is, the optimal plan is regular.

From an economic point of view, $\overline{s_0}$ acts as a threshold. If the initial inventory s_0 is greater than $\overline{s_0}$, then holding such a (high) amount of stock is too expensive. It is then optimal for the firm to sell out immediately the quantity $\alpha = s_0 - \overline{s_0}$, so as to reduce its initial inventory to $\overline{s_0}$. This is the maximal level that it can afford to hold. Recall that the quantity α is sold at an infinite rate and hence at the lowest price $\dot{\pi}(\infty)$. Therefore the firm has no interest in selling more than necessary. This explain why it sells exactly $s_0 - \overline{s_0}$.

For every $s_0 > \overline{s_0}$, the regular part (x, y) of the optimal plan is the same, it corresponds to the optimal plan obtained for an initial inventory level equal to $\overline{s_0}$. Only the size of the jumps on the sales $\alpha = s_0 - \overline{s_0}$ changes with s_0. If $s_0 \leq \overline{s_0}$ then, the optimal plan is regular and actually depends on s_0 since it is obtain from the translation of w^a on the interval $(0, T - \tau_0]$ where τ_0 solves $s_0 = \int_{\tau_0}^T \delta(w^a(u))du$.

Remark that the existence of such a threshold and the corresponding properties of the optimal plan will also hold in the case of a short planning period. We shall determine this threshold but not reproduce the discussion.

To the contrary what follows is specific to the long planning periods ($T > T_a$). First, the threshold $\overline{s_0}$ does not depend on T. Second, the plan that the firm adopts is in two (non trivial) phases. The first phase consists in selling out the initial inventory in an optimal way. During the second phase, the firm follows the just-in-time strategy, producing and selling at the same constant rate a, until the end of the period. The depletion phase does not depend on T. It is the one associated to the optimal plan obtained on the period $[0, T_a]$. In view of item 2, this property is straightforward when $s_0 > \overline{s_0} = \int_{\tau(a)}^T \delta(w^a(u))du$. When $s_0 \leq \overline{s_0}$, it follows from Remark 11.

We end up this first step of the constructive resolution by stressing the fact that, in the case of a long planning period, nothing more than the computation of w^a is necessary to solve the planning problem. Let us now turn to the short planning period case.

3.2 The Short Planning Period Case

In this section we assume that $\tau(a) \le 0$. By using Remark 10 and the Mean-Value Theorem, we see that, if $0 < s_0 \le \int_0^T \delta(w^a(s))ds = S(0,a)$ then, there exists some $\tau_0 \in [0,T)$ such that $s_0 = \int_0^T \delta(w^a(s))ds$. Therefore the answer to (27) is affirmative and the optimal plan is obtained as previously by using Remark 9. We have :

Theorem 13 *Assume that $\tau(a) \le 0$.*

If $s_0 \le \int_0^T \delta(w^a(u))du$ then, there exists a unique $\tau_0 \in [0,T)$ such that $\int_{\tau_0}^T \delta(w^a(u))du$ $= s_0$ and the optimal plan is regular given by :

$$\begin{pmatrix} \alpha \\ x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ w^a(\cdot + \tau_0)\mathbf{1}_{(0,T-\tau_0]} + a\mathbf{1}_{(T-\tau_0,T]} \\ g\left(w^a(\cdot + \tau_0)\mathbf{1}_{(0,T-\tau_0]} + a\mathbf{1}_{(T-\tau_0,T]}\right) \end{pmatrix}$$

From Theorem 13 we see that even if the planning period is short, when the initial inventory s_0 is lower than $\int_0^T \delta(w^a(u))du$, the firm has time to clear out its stock before the horizon T, selling out at the rate $x = w^a(\cdot + \tau_0)$ on $(0, T-\tau_0]$ and it can afford a static management final phase : $x = y = a$ on $(\tau_0, T]$. We shall find out that this is not the case when the initial inventory is greater than $\int_0^T \delta(w^a(u))du$.

Assume that $s_0 > \int_0^T \delta(w^a(s))ds = S(0,a)$. Since the function $t \longmapsto S(t,a) = \int_t^T \delta(w^a(u))du$ is decreasing, if there exists some $\tau \in [\tau(a), T)$ such that $\int_\tau^T \delta(w^a(s))ds = s_0$ then, $\tau < 0$. Therefore the answer to (27) is negative. In that case, the planning problem can not be solved by using w^a. One has to consider the solutions w^r of $BW(r)$ for $r > a$ and hence, by condition (7) of Corollary 6, which are defined at least on $(0,T]$, i.e. such that $\tau(r) \le 0$.

The planning problem can be solved by finding $\hat{r} \in \{r \in [a,\infty) \mid \tau(r) \le 0\}$ such that

$$\begin{array}{lll} \text{(a)} & \text{either} & \int_0^T \delta(w^{\hat{r}}(u))du = s_0, \\ \text{(b)} & \text{or} & \int_0^T \delta(w^{\hat{r}}(u))du < s_0 \text{ and } w^{\hat{r}}(0+) = \infty. \end{array} \tag{28}$$

Indeed, we have the following

Remark 14 *If $\hat{r} \in \tau^{-1}(\mathbb{R}^-) \triangleq \{r \in [a,\infty) \mid \tau(r) \le 0\}$ is such that (28) holds then, by Theorem 7, $w^{\hat{r}}$ satisfies the requirements of Corollary 6 with $(\tau_0, r) = (0, \hat{r})$. The optimal plan is then given by (11) in function of $w^{\hat{r}}$ which reads*

$$(\alpha, x, y) = \begin{cases} (0, w^{\hat{r}}, g(w^{\hat{r}})) & \textit{if} \quad \textit{(28a) holds,} \\ (s_0 - \int_0^T \delta(w^{\hat{r}}(u))du, w^{\hat{r}}, g(w^{\hat{r}})) & \textit{if} \quad \textit{(28b) holds.} \end{cases} \tag{29}$$

We therefore now focus our attention on the function

$$\begin{aligned} S_0 \ : \ \tau^{-1}(\mathbb{R}^-) = \{r \in [a,\infty) \mid \tau(r) \le 0\} \quad &\longrightarrow \quad \mathbb{R}^+ \cup \{\infty\} \\ r \quad &\longmapsto \quad S_0(r) = \int_0^T \delta(w^r(s))ds. \end{aligned}$$

It represents the level at time 0+ of the backward cumulative inventory process associated to any w^r solution of $BW(r)$ defined at least on $(0, T]$. In particular, we shall investigate the monotony and continuity of the function S_0 with respect to r in order to find the greatest value that it can reach when the terminal condition r varies in $\tau^{-1}(\mathbb{R}^-)$. This will allow us to determine the threshold $\overline{s_0}$ below (above) which the optimal plan is (not) regular. For this purpose we work under the following technical

Standing Assumption
(Hs) *The function s is strictly convex and $\dot{s}$ is locally Lipschitz on $\mathbb{R}^+$.*

Proposition 15 *The function $\tau \ : \ r \in [a,\infty) \longmapsto \tau(r) \in [-\infty, T)$ is nondecreasing, right-continuous and satisfies $\lim_{r\to\infty} \tau(r) = T$. Therefore, if the set $\tau^{-1}(\mathbb{R}^-)$ is not empty then, it is a bounded interval with $\min \tau^{-1}(\mathbb{R}^-) = a$. We set $m \triangleq \sup\{\tau^{-1}(\mathbb{R}^-)\} < \infty$.*

Proposition 16 *Assume that $\tau^{-1}(\mathbb{R}^-) \neq \emptyset$.*

1. *The function S_0 is well-defined, nondecreasing and left-continuous as a map from $\tau^{-1}(\mathbb{R}^-)$ into $\mathbb{R}^+ \cup \{\infty\}$.*
2. *The function S_0 is continuous on its domain $\mathrm{Dom}(S_0) \triangleq \{r \in \tau^{-1}(\mathbb{R}^-) \mid S_0(r) \in \mathbb{R}^+\}$. If $\tau(r) < 0$ then, $r \in \mathrm{Dom}(S_0)$.*
3. *If $\tau(m) > 0$ i.e. $\tau^{-1}(\mathbb{R}^-) = [a, m)$ then, $\sup_{\tau^{-1}(\mathbb{R}^-)} S_0 = \infty$.*

Corollary 17 *1. The function S_0 has at most one discontinuity as a map from $\tau^{-1}(\mathbb{R}^-)$ into $\mathbb{R}^+ \cup \{\infty\}$. If $d \in \tau^{-1}(\mathbb{R}^-)$ is such a point of discontinuity then, $d < m$, $\tau(d) = 0$, $\mathrm{Dom}(S_0) = [a, d]$ and hence,*

$$\max_{\mathrm{Dom}(S_0)} S_0 = S_0(d) = \int_0^T \delta(w^d(u))du < \infty \quad \text{and} \quad \inf_{\substack{r>d \\ r\in\tau^{-1}(\mathbb{R}^-)}} S_0(r) = \infty \ .$$

2. If $\sup_{\tau^{-1}(\mathbb{R}^-)} S_0 < \infty$ then, $\tau(m) = 0$, and hence, $\mathrm{Dom}(S_0) = \tau^{-1}(\mathbb{R}^-) = [a, m]$ and

$$\max_{\mathrm{Dom}(S_0)} S_0 = \max_{\tau^{-1}(\mathbb{R}^-)} S_0 = S_0(m) = \int_0^T \delta(w^m(u))du < \infty \ .$$

From Corollary 17 we see that, as for continuity and boundness of S_0, there are exactly three cases to keep distinct. The function S_0 may be continuous and unbounded on $\tau^{-1}(\mathbb{R}^-)$, continuous and bounded on $\tau^{-1}(\mathbb{R}^-)$ or it may have a unique discontinuity at some point $d \in \tau^{-1}(\mathbb{R}^-)$. In addition,

$$\sup_{\mathrm{Dom}(S_0)} S_0 = \begin{cases} \infty & \text{if } S_0 \text{ is continuous and unbounded,} \\ S_0(m) = \int_0^T \delta(w^m(u))du < \infty & \text{if } S_0 \text{ is continuous and bounded,} \\ S_0(d) = \int_0^T \delta(w^d(u))du < \infty & \text{if } S_0 \text{ has a discontinuity at } d \in \tau^{-1}(\mathbb{R}^-)\,. \end{cases}$$

By continuity of S_0 on its domain, we shall obtain that the threshold $\overline{s_0}$ on the initial inventory below (above) which the optimal plan is (not) regular exists and is given by

$$\overline{s_0} = \sup_{\mathrm{Dom}(S_0)} S_0\,.$$

Theorem 18 *Assume that $\tau(a) \le 0$ and that S_0 is continuous and unbounded on $\tau^{-1}(\mathbb{R}^-)$. Consider $s_0 > \int_0^T \delta(w^a(u))du$. There exists some $\hat{r} \in \tau^{-1}(\mathbb{R}^-)$, $\hat{r} > a$, such that $\int_0^T \delta(w^{\hat{r}}(u))du = s_0$ and the optimal plan is regular, given by*

$$(\alpha, x, y) = (0, w^{\hat{r}}, g(w^{\hat{r}}))\,.$$

Proof. Since S_0 is continuous and unbounded on the interval $\tau^{-1}(\mathbb{R}^-)$ and $s_0 > \int_0^T \delta(w^a(u))du = S_0(a) = \min_{\tau^{-1}(\mathbb{R}^-)} S_0$, by the Mean-Value Theorem there exists some $\hat{r} \in \tau^{-1}(\mathbb{R}^-)$ such that $S_0(\hat{r}) = \int_0^T \delta(w^{\hat{r}}(u))du = s_0$ ($\hat{r} > a$ since $s_0 > S_0(a)$). This means that condition (28a) holds for $\hat{r}$. The proof is completed by using Remark 14. □

Theorem 19 *Assume that $\tau(a) \le 0$ and that S_0 is continuous and bounded on $\tau^{-1}(\mathbb{R}^-)$. Consider $s_0 > \int_0^T \delta(w^a(u))du$.*

1. *If $s_0 \le S_0(m) = \int_0^T \delta(w^m(u))du$ then, there exists some $\hat{r} \in (a, m]$ such that $\int_0^T \delta(w^{\hat{r}}(u))du = s_0$ and the optimal plan is regular, given by*

$$(\alpha, x, y) = (0, w^{\hat{r}}, g(w^{\hat{r}}))\,.$$

2. *If $s_0 > S_0(m) = \int_0^T \delta(w^m(u))du$ then, the optimal plan is not regular, given by*

$$(\alpha, x, y) = (s_0 - \int_0^T \delta(w^m(u))du, w^m, g(w^m))\,.$$

Proof. By 2 of Corollary 17, if S_0 is bounded on $\tau^{-1}(\mathbb{R}^-)$ then $\tau(m) = 0$, $\tau^{-1}(\mathbb{R}^-) = [a, m]$ and $\max_{\tau^{-1}(\mathbb{R}^-)} S_0 = S_0(m)$. Therefore, if in addition S_0 is continuous on $\tau^{-1}(\mathbb{R}^-)$ then, by the Mean-Value Theorem there exists some $\hat{r} \in (a, m]$ such that $\int_0^T \delta(w^{\hat{r}}(u))du = s_0$ and then, item 1 holds by Remark 14. Now, assume that $s_0 > S_0(m) = \int_0^T \delta(w^m(u))du$. Since $\tau(m) = 0 > -\infty$, Proposition 7 implies that $w^m(0+) = w^m(\tau(m)+) = \infty$. Therefore condition (28b) holds with $\hat{r} = m$ and the proof is concluded by using Remark 14. □

Theorem 20 *Assume that $\tau(a) \leq 0$ and that S_0 has a (unique) discontinuity at $d \in \tau^{-1}(\mathbb{R}^-)$. Consider $s_0 > \int_0^T \delta(w^a(u))du$.*

1. *If $s_0 \leq S_0(d) = \int_0^T \delta(w^d(u))du$ then, there exists some $\hat{r} \in (a, d]$ such that $\int_0^T \delta(w^{\hat{r}}(u))du = s_0$ and the optimal plan is regular, given by*

$$(\alpha, x, y) = (0, w^{\hat{r}}, g(w^{\hat{r}})) .$$

2. *If $s_0 > S_0(d) = \int_0^T \delta(w^d(u))du$ then, the optimal plan is not regular, given by*

$$(\alpha, x, y) = (s_0 - \int_0^T \delta(w^d(u))du, w^d, g(w^d)) .$$

Proof. By 1 of Corollary 17, if S_0 has a discontinuity at some $d \in \tau^{-1}(\mathbb{R}^-)$ then $\tau(d) = 0$, $\mathrm{Dom}(S_0) = [a, d]$ and $\max_{\mathrm{Dom}(S_0)} S_0 = S_0(d) < \infty$. Since by Proposition 16 S_0 is continuous on its domain, item 1 is obtained as above by using the Mean-Value Theorem and Remark 14. Item 2 is also obtained by the same arguments as above with d instead of m. $\square$

We have seen from Theorem 13, that when $s_0 \leq \int_0^T \delta(w^a(u))du$ the firm can afford to sell out its inventory at some rate which ranges a part of the path of w^a. From Theorems 18, 19 and 20 we see that if the initial inventory is higher than $\int_0^T \delta(w^a(u))du$ then, the firm must move out its stock faster : the optimal sales rate is given by $x = w^{\hat{r}}$ on $(0, T]$ with $\hat{r} > a$ so that it is greater than w^a on the whole planning period (see Proposition 26 in Appendix B for the increasing feature of w^r with respect to r). In this context, the production may never start. Typically, if $\dot{c}(0) > \dot{\pi}(\infty)$ and $\hat{r} > \dot{\pi}^{-1}(\dot{c}(0))$ then, $\dot{\pi}(x) = \dot{\pi}(w^{\hat{r}}) \leq \dot{\pi}(\hat{r}) < \dot{c}(0)$ on $(0, T]$ and hence by definition of g, $y = 0$ on $(0, T]$. In any case, the inventory is totally depleted right in T, there is no phase where the firm produces for immediate sales at the constant rate a. This makes qualitatively different the optimal plan obtained on a short planning period and the one obtained on a long period.

From Theorems 13, 18, 19 and 20 we check that there is positive threshold above (below) which the optimal plan is with (without) depletion at time 0. It is given by

$$\overline{s_0} \triangleq \sup_{\mathrm{Dom}(S_0)} S_0 = \sup_{\{r \in [a,\infty) | \tau(r) \leq 0 \,, \int_0^T \delta(w^r(u))du < \infty\}} \int_0^T \delta(w^r(u))du \,.$$

Recall that by time homogeneity of BW (see Remark 11) we have observe that the length T_a of the domain of the maximal solution of $BW(a)$ and the threshold $\overline{s}_0$ obtained for long planning periods ($T > T_a$) do not depend on T. To the contrary, when the planning period is short ($T < T_a$), the threshold may depend on T. Let us denote $(w^r_{T_a}, \tau_{T_a}(r))$ the maximal solution of $BW(r)$ with terminal time T_a, for any $r \in [a, \infty)$. By using Remark

11 we see that, given π, c, s and hence T_a, the dependence of $\overline{s_0}$ with respect to T is given by :

$$\overline{s_0}(T) = \sup_{\{r\in[a,\infty)|\tau_{T_a}(r)\leq 0\,,\int_{T_a-T}^{T_a}\delta(w^r_{T_a}(u))du<\infty\}} \int_{T_a-T}^{T_a} \delta(w^r_{T_a}(u))du$$

It would be of interest to study the monotony of $\overline{s_0}$ with respect to T and it would be economically funded to obtain that the shorter the period is, the lower the threshold is. Also, notice that, while we indeed prove the existence of such a threshold, we do not provide some conditions for deciding whether if it is infinite or finite, with S_0 continuous or not. These are directions for future research.

Appendix A : Proof of Theorem 7.

Let us denote

$$D = (\dot{\pi}(\infty), \dot{\pi}(a)].$$

By Assumption (H) we can define Δ on D by

$$\Delta(v) \triangleq \delta(\dot{\pi}^{-1}(v)) = \dot{\pi}^{-1}(v) - \dot{c}^{-1}(v)\mathbf{1}_{v>\dot{c}(0)}, \ \ \forall v \in D.$$

For later purpose, we make the following

Remark 21 *By Assumptions (H), $(H\pi)$ and (Hc), Δ is nonnegative, locally Lipschitz and bounded from above by $\dot{\pi}^{-1}$ on D.*

We shall obtain Theorem 7 as a Corollary of

Theorem 7bis *For all $\theta \in D$, there exists a unique couple $(\tau, z) \in [-\infty, T) \times C((\tau, T]; D)$, such that z satisfies the following equation on $(\tau, T]$*

$$\widetilde{BW}(\theta) \ \ : \ \ z(t) = \theta - \int_t^T \left[\lambda z(u) + \dot{s}\left(\int_u^T \Delta(z(s))ds\right)\right] du,$$

and such that,

$$\text{either } \tau = -\infty \ , \quad \text{or, } \tau > -\infty \text{ and } z(\tau+) \triangleq \lim_{t\searrow\tau} z(t) = \dot{\pi}(\infty). \tag{30}$$

Moreover, z is increasing.

If Theorem 7bis holds then, for all $r \in [a, \infty)$, since $\dot{\pi}(r) \in D$, there exists a unique couple $(\tau, z) \in [-\infty, T) \times C((\tau, T]; D)$ such that

$$z(t) = \dot{\pi}(r) - \int_t^T \left[\lambda z(u) + \dot{s}\left(\int_u^T \Delta(z(s))ds\right)\right] du \ , \ \ \forall t \in (\tau, T] \tag{31}$$

and such that the boundary condition (30) holds. Since z is continuous on $(\tau, T]$, by Remark 21 and by Assumption (Hs), we see that the function

$$u \longrightarrow \lambda z(u) + \dot{s}\left(\int_u^T \Delta(z(s))ds\right)$$

is continuous on $(\tau, T]$. We then deduce from (31) that z is in actual fact continuously differentiable on $(\tau, T]$ with

$$\dot{z}(t) = \lambda z(t) + \dot{s}\left(\int_t^T \Delta(z(s))ds\right), \ \forall t \in (\tau, T]. \tag{32}$$

Then, it is easy to check that z satisfies

$$e^{-\lambda t} z(t) = e^{-\lambda T}\dot{\pi}(r) - \int_t^T e^{-\lambda u}\ \dot{s}\left(\int_u^T \Delta(z(s))ds\right) du, \ \forall t \in (\tau, T]. \tag{33}$$

Under Assumption (H), we can set $w \triangleq \dot{\pi}^{-1} \circ z$ and see that, since z is in $C((\tau, T]; D)$, w is in $C((\tau, T]; [a, \infty))$. From (32) and (30), we deduce that w solves $BW(r)$ on $(\tau, T]$ and satisfies the following boundary condition

$$\text{either} \quad \tau = -\infty, \quad \text{or,} \quad \tau > -\infty \ \text{ and } \ w(\tau+) \triangleq \lim_{t \searrow \tau} w(t) = \infty.$$

Besides, since z is increasing, it follows from assumption (H) again, that w is decreasing. This concludes the proof of Theorem 7.

We now turn to the proof of Theorem 7bis. It is based on the three following propositions and an induction argument. More precisely, some regularity and Lipschitzianity properties of the right-hand side of the equation $\widetilde{BW}(\theta)$ are given in Proposition 22. Using this properties we will prove that for every terminal condition $\theta \in D$, the equation $\widetilde{BW}(\theta)$ has at least one solution : there exists some continuous function z which satisfies $\widetilde{BW}(\theta)$ on some non-empty interval $(\gamma, T]$. We will also provide some result of continuous dependence of solutions with respect to the terminal condition. This is Proposition 23. With Proposition 24, we will see that a solution (γ, z) which does not satisfy the boundary condition (30) can be extended in another solution on some larger interval.

In the sequel, for $\phi \in C([a, b])$, we denote $\|\phi\|_{[a,b]} \triangleq \sup_{a \le s \le b} |\phi(s)|$.

The right-hand side of the backward equation is studied in terms of the following operator : for fixed $\eta \in [-\infty, T)$,

$$\begin{aligned} F_\eta \ : \ (\eta, T] \times C((\eta, T]; D) \quad &\longrightarrow \quad \mathbb{R} \\ (t, v) \quad &\longmapsto \quad \lambda v(t) + \dot{s}\left(\int_t^T \Delta(v(s))ds\right) \end{aligned}$$

Proposition 22 *Let $\eta \in [-\infty, T)$ and let L be some compact subset of D.*

(o) *The operator F_η is well defined and takes values in $(0, \infty)$.*

(i) *For all $v \in C((\eta, T]; D)$, the function $t \longrightarrow F_\eta(t, v)$ is continuous on $(\eta, T]$.*

(ii) *There exists some constant $K_{\eta,L} > 0$ such that, for all $t \in (\eta, T]$,*

$$v\ ,\ \tilde{v}\ \in C([t, T]; L) \implies |F_\eta(t, v) - F_\eta(t, \tilde{v})| \leq K_{\eta,L} \|v - \tilde{v}\|_{[t,T]},$$

Proof. Let $v \in C((\eta, T]; D)$. First, since $D = (\dot{\pi}(\infty), \dot{\pi}(a)] \subset (0, \infty)$, v takes values in $(0, \infty)$. Second, from Remark 21, $\Delta \circ v$ is continuous and nonnegative on $(\eta, T]$. This implies that the function $t \longmapsto \int_t^T \Delta(v(s))ds$ is well-defined, taking values in $\mathbb{R}^+$ and continuous on $(\eta, T]$. Since $\dot{s}$ is continuous on $\mathbb{R}^+$, (o) and (i) of Proposition 22 hold. We now prove (ii).

Let $t \in (\eta, T]$. Notice that, by Remark 21,

$$0 \leq \Delta(u) \leq \dot{\pi}^{-1}(u),\ \forall u \in D.$$

Therefore, since $\dot{\pi}^{-1}$ is decreasing on D, and v, $\tilde{v}$ are taking values in $L \subset D$, we have

$$\int_t^T \Delta(v(s))ds\ ,\ \int_t^T \Delta(\tilde{v}(s))ds\ \in [0, (T - \eta)\dot{\pi}^{-1}(\min L)].$$

It then follows from (Hs) that there exists some constant $C_{\eta,L} > 0$ such that

$$\begin{aligned} |F_\eta(t, v) - F_\eta(t, \tilde{v})| &\leq \lambda |v(t) - \tilde{v}(t)| + \left| \dot{s}\left(\int_t^T \Delta(v(s))ds \right) - \dot{s}\left(\int_t^T \Delta(\tilde{v}(s))ds \right) \right| \\ &\leq \lambda |v(t) - \tilde{v}(t)| + C_{\eta,L} \left| \int_t^T \Delta(v(s))ds - \int_t^T \Delta(\tilde{v}(s))ds \right|. \end{aligned}$$

By Remark 21, it follows from the last inequality that there exists some constant $K_L > 0$ such that

$$|F_\eta(t, v) - F_\eta(t, \tilde{v})| \leq \lambda |v(t) - \tilde{v}(t)| + K_L C_{\eta,L} \int_t^T |v(s) - \tilde{v}(s)| ds, \tag{34}$$

and hence $|F_\eta(t, v) - F_\eta(t, \tilde{v})| \leq K_{\eta,L} \|v - \tilde{v}\|_{[t,T]}$, for some constant $K_{\eta,L}$. This ends the proof of Proposition 22. $\square$

Observe that if $\eta_1 \leq \eta_2$ then, $F_{\eta_1 |(\eta_2,T] \times C((\eta_2,T];D)} \equiv F_{\eta_2}$. Therefore, in the sequel we shall omit the index and always consider F with its maximal set of definition.

Definition 1 *Given $\theta \in D$, we shall say that (γ, z) is a solution of $\widetilde{BW}(\theta)$ if :*
$\gamma \in [-\infty, T)$, $z \in C((\gamma, T]; D)$ and z satisfies the equation $\widetilde{BW}(\theta)$ on $(\gamma, T]$.

Recall that the equation $\widetilde{BW}(\theta)$ reads as follows in terms of F :

$$\widetilde{BW}(\theta)\ :\ z(t) = \theta - \int_t^T F(u, z)du,\ \ t \leq T.$$

Proposition 23

1. For all $\theta \in D$, the equation $\widetilde{BW}(\theta)$ has at least one solution. If (γ, z) is such a solution then, z is increasing on $(\gamma, T]$.

2. Let $\eta \in (-\infty, T]$ and let L be a compact subset of D. Then, there exists some constant $K_{\eta,L} > 0$ such that :

if (γ, z) is a solution of $\widetilde{BW}(\theta)$ and (γ', z') is a solution of $\widetilde{BW}(\theta')$ then, for all $t \in [\eta, T]$ such that z and z' are both defined on $[t, T]$ and both map $[t, T]$ into L we have

$$\|z - z'\|_{[t,T]} \leq K_{\eta,L}|\theta - \theta'| \ .$$

Proof. We begin with the

Proof of item 1. Let $\theta \in D$. First observe that the last assertion is a direct consequence of the nonnegativity of $\dot{s}$ and the positivity of elements of D. We now concentrate on the first assertion. We shall use a Picard's approximations argument.

Step 1. We start with the construction of a suitable set in which we will construct our Picard's approximations. For this purpose, let us fix some $\eta \in (-\infty, T)$ and choose some $b \geq 0$ such that the compact

$$L \triangleq [\theta - b, \min\{(\theta + b), \dot{\pi}(a)\}] \text{ is included in } D. \tag{35}$$

We claim that there exists some constant $M > 0$ for which

$$\forall t \in (\eta, T] \ , \ \ \forall v \in C([t, T]; D) \ \ : \qquad \|v - \theta\|_{[t,T]} \leq b \ \implies \ |F(t, v)| \leq M. \tag{36}$$

To see this, first notice that by Proposition 22 (o) and (i), the function $t \longrightarrow F(t, \theta)$ is continuous on $(-\infty, T]$, with positive values and therefore bounded on $[\eta, T]$ by some constant $M_0 > 0$. Let us now prove that (36) holds for $M = M_0 + K_{\eta,L}$ where $K_{\eta,L}$ is given by Proposition 22 (ii).

Let $t \in (\eta, T]$ and $v \in C([t, T]; D)$ with $\|v - \theta\|_{[t,T]} \leq b$. Then, by definition of L, v is in $C([t, T]; L)$. Since the taking values in L, we therefore have by Proposition 22 (ii)

$$|F(t, v) - F(t, \theta)| \leq K_{\eta,L}\|v - \theta\|_{[t,T]} \leq K_{\eta,L}b$$

and hence, $|F(t, v)| \leq K_{\eta,L}b + M_0 \triangleq M$. This provides (36).

We are now in position to construct a suitable set. Recalling that $\eta < T$, we may choose some $\gamma \in (\eta, T)$ such that

$$0 < T - \gamma < \min\{T - \eta, \frac{b}{M}, \frac{1}{K_{\eta,L}}\}. \tag{37}$$

Our Picard's approximations will be constructed on the set

$$\mathcal{S} \triangleq \{ \ v \in C([\gamma, T]; L) \mid v(T) = \theta, \ |v(t) - v(u)| \leq M|t - u|, \forall t, u \in [\gamma, T] \ \}.$$

Observe that $\mathcal{S}$ is closed for the pointwise convergence on $[\gamma, T]$.
We now define an operator P on $\mathcal{S}$ by

$$P(v)(t) \triangleq \theta - \int_t^T F(u,v)du, \quad \forall\, t \in [\gamma, T]\ .$$

We shall prove in the following steps that P admits a fixed point in $\mathcal{S}$.

Step 2. We first prove that P maps $\mathcal{S}$ into $\mathcal{S}$. Let $v \in \mathcal{S}$. First recall that by Proposition 22 (o) we have

$$F(u,v) > 0\ , \quad \forall u \in [\gamma, T] \tag{38}$$

and notice that by (36), (37) and by definition of $\mathcal{S}$ we have

$$F(u,v) \leq M\ , \quad \forall u \in [\gamma, T]. \tag{39}$$

We now check that $P(v)$ takes values in L.
Let $t \in [\gamma, T]$. By equations (38) and (39) and by definition of γ (see (37)), we have

$$P(v)(t) = \theta - \int_t^T F(u,v)du \leq \theta \leq \min\{(\theta + b), \dot{\pi}(a)\},$$

and

$$P(v)(t) = \theta - \int_t^T F(u,v)du \geq \theta - M(T-\gamma) \geq \theta - b$$

which implies that $P(v)(t) \in L$. The proof of Step 2 is concluded by writing

$$|P(v)(t) - P(v)(u)| = \left| \int_u^t F(s,v)ds \right| \leq M|t-u|,\ \forall t, u \in [\gamma, T],$$

where the last inequality follows from (39).

Step 3. We shall now construct a Cauchy sequence in $\mathcal{S}$ that converges to a fixed point of P. Let $(z_n)_n$ be the sequence defined by

$$z_0 = \theta \text{ on } [\gamma, T] \ \text{ and } \ z_{n+1} = P(z_n), \quad \forall n \in \mathbb{N}.$$

Since $\gamma \in (\eta, T]$, it follows from Proposition 22 (ii) that, for all $t \in [\gamma, T]$ we have

$$\begin{aligned} |z_{n+1}(t) - z_n(t)| &= \left| \int_t^T [F(u, z_n) - F(u, z_{n-1})]du \right| \\ &\leq |T-t| K_{\eta,L} \|z_n - z_{n-1}\|_{[t,T]} \\ &\leq ((T-\gamma)K_{\eta,L})^n \|z_1 - z_0\|_{[\gamma,T]}. \end{aligned}$$

By (37), $0 < (T-\gamma)K_{\eta,L} < 1$. Therefore, (z_n) is a Cauchy sequence. Since $\mathcal{S}$ is closed, its limit z is in $\mathcal{S}$. Arguing as above, we see that

$$\|P(z) - P(z_n)\|_{[\gamma,T]} \leq K_{\eta,L}(T-\gamma)\|z - z_n\|_{[\gamma,T]}.$$

It follows that

$$\begin{aligned}\|P(z)-z\|_{[\gamma,T]} &= \|P(z)-P(z_n)\|_{[\gamma,T]}+\|z_{n+1}-z\|_{[\gamma,T]}\\ &\le K_{\eta,L}(T-\gamma)\|z-z_n\|_{[\gamma,T]}+\|z_{n+1}-z\|_{[\gamma,T]},\end{aligned}$$

which shows, by passing to the limit, that $P(z)=z$ on $[\gamma,T]$. This completes the proof of the first item of Proposition 23. We now turn to the

Proof of item 2. Let $t\in[\eta,T]$ such that z and z' both map $[t,T]$ into L. Then, for all $u\in[t,T]$, z and z' both map $[u,T]$ into L. Therefore, by (34) in the proof of Proposition 22, there exists some $C=C_{\eta,L}>0$ such that

$$|F(u,z)-F(u,z')|\le\lambda|z(u)-z'(u)|+C\int_u^T|z(v)-z'(v)|dv,\ \forall u\in[t,T].$$

This implies that, for all s in $[t,T]$, we have

$$\begin{aligned}|z(s)-z'(s)| &\le |\theta-\theta'|+\int_s^T|F(u,z)-F(u,z')|du\\ &\le |\theta-\theta'|+\int_s^T\left(\lambda|z(u)-z'(u)|+C\int_u^T|z(v)-z'(v)|dv\right)du\\ &\le |\theta-\theta'|+(\lambda+C(T-\eta))\int_s^T|z(u)-z'(u)|du\\ &\le |\theta-\theta'|+C_{\eta,L}\int_s^T|z(u)-z'(u)|du.\end{aligned}$$

By applying Gronwall's Lemma to the last inequality and since $\eta\le s$ we obtain

$$|z(s)-z'(s)| \le |\theta-\theta'|e^{C_{\eta,L}(T-s)}\le|\theta-\theta'|e^{C_{\eta,L}(T-\eta)},$$

which ends the proof of Proposition 23. □

We now turn to the extension Proposition.

Proposition 24 *Let (γ,z) be a solution of $\widetilde{BW}(\theta)$ for some $\theta\in D$. If $\gamma>-\infty$ and if z satisfies*

$$z(\gamma+)\triangleq\lim_{t\searrow\gamma}z(t)>\dot\pi(\infty)$$

then, $\widetilde{BW}(\theta)$ has a solution $(\hat\gamma,\hat z)$ which satisfies : $\hat\gamma<\gamma$ and $\hat z_{|(\gamma,T]}\equiv z$.

Proof. Let (γ,z) be a solution of $\widetilde{BW}(\theta)$ satisfying $\gamma>-\infty$. Since z is increasing the limit $z(\gamma+)=\lim_{t\searrow\gamma}z(t)$ exists in $\overline{D}=[\dot\pi(\infty),\dot\pi(a)]$. Let us assume that $z(\gamma+)>\dot\pi(\infty)$. We have to prove that there exists some $\hat\gamma<\gamma$ and some $\hat z\in C((\hat\gamma,T];D)$ such that $\hat z$ satisfies $\widetilde{BW}(\theta)$ on $(\hat\gamma,T]$ i.e.

$$\hat z(t)=\theta-\int_t^T\left[\lambda\hat z(u)+\dot s\left(\int_u^T\Delta(\hat z(s))ds\right)\right]du\ ,\ \ \forall t\in(\hat\gamma,T],$$

and such that

$$\hat{z}_{|(\gamma,T]} \equiv z$$

To do this, recall that z satisfies

$$z(t) = \theta - \int_t^T \left[\lambda z(u) + \dot{s}\left(\int_u^T \Delta(z(s))ds\right)\right] du\ ,\ \ \forall t \in (\gamma, T],$$

and hence, by passing to the limit as t goes to γ

$$z(\gamma+) = \theta - \int_\gamma^T \left[\lambda z(u) + \dot{s}\left(\int_u^T \Delta(z(s))ds\right)\right] du.$$

Therefore, it suffices to prove that

$$\left\{ \begin{array}{l} \text{there exists some } \hat{\gamma} < \gamma \text{ and some } \tilde{z} \in C((\hat{\gamma}, \gamma]; D) \text{ such that} \\ \tilde{z}(t) = z(\gamma+) - \displaystyle\int_t^\gamma \left[\lambda \tilde{z}(u) + \dot{s}\left(\int_u^\gamma \Delta(\tilde{z}(s))ds + \int_\gamma^T \Delta(z(s))ds\right)\right] du\ ,\ \ \forall t \in (\hat{\gamma}, \gamma]. \end{array} \right. \tag{40}$$

Indeed, the result will then hold for $(\hat{\gamma}, \hat{z})$ where $\hat{z}$ is defined by setting $\hat{z}_{|(\hat{\gamma},\gamma]} \equiv \tilde{z}$ and $\hat{z}_{|(\gamma,T]} \equiv z$.
Let us establish (40). For ease of notation, let us write $I = \int_\gamma^T \Delta(z(s))ds$. Now, for all $\eta \in [-\infty, \gamma]$ consider the operator defined by

$$\begin{array}{rcl} \tilde{F}_\eta\ :\ (\eta, \gamma] \times C((\eta, \gamma]; D) & \longmapsto & \mathbb{R} \\ (t, v) & \longrightarrow & \lambda v(t) + \dot{s}\left(\displaystyle\int_t^T \Delta(v(s))ds + I\right) \end{array}$$

Since the translation $\dot{s}(.+I)$ inherits of the increasing and Lipschitzian feature of $\dot{s}$, one should be convinced that $\tilde{F}_\eta$ satisfies the regularity and Lipschitzianity properties gathered in Proposition 22. This properties together with the fact that $z(\gamma+)$ is in D are sufficient to let the Picard's approximations argument be valid to prove the existence of some $\hat{\gamma} < \gamma$ and some $\tilde{z} \in C((\hat{\gamma}, \gamma]; D)$ such that $\tilde{z}$ satisfies (40) on $(\hat{\gamma}, \gamma]$. This ends the proof of Proposition 24. □

We are now in position to give the

Proof of Theorem 7bis.

Let us first establish the uniqueness. Let (γ, z) and (γ', z') satisfying the requirements of Theorem 7bis. Fix some arbitrary $t \in (\max(\gamma, \gamma'), T]$. Since z and z' are in $C((\max\{\gamma, \gamma'\}, T]; D)$, there exists some compact $L \subset D$ such that z and z' both map $[t, T]$ into L. It follows from item 2 of Proposition 23 that $z \equiv z'$ on $[t, T]$. By arbitrariness of t in $(\max(\gamma, \gamma'), T]$, we then have $z \equiv z'$ on $(\max(\gamma, \gamma'), T]$, so that $\gamma,\ \gamma' \geq \max\{\gamma, \gamma'\}$. Therefore $\gamma = \gamma'$ and hence $(\gamma, z) = (\gamma, z')$. This provides uniqueness.

We now prove that existence holds. Define

$$E \triangleq \{(\gamma, z) \text{ solution of } \widetilde{BW}(\theta)\}.$$

By item 1 of Proposition 23, E is not empty. It is easy to check that E is inductive for the order defined by

$$(\gamma_2, z_2) \succeq (\gamma_1, z_1) \text{ iff } \gamma_2 \leq \gamma_1 \text{ and } z_2 \equiv z_1 \text{ on } (\gamma_1, T].$$

Thus, by Zorn's lemma, it admits a maximal element (τ, z). If $\tau = -\infty$, then the proof is concluded. We now assume that $\tau \neq -\infty$. We have to prove that $z(\tau+) = \dot{\pi}(\infty)$. Assume to the contrary that $z(\tau+) > \dot{\pi}(\infty)$. Then, by Proposition 24 there exists some $(\hat{\tau}, \hat{z})$ in E such that $\hat{\tau} < \tau$ and $\hat{z}_{|(\tau,T]} \equiv z$. This contradicts the maximal feature of (τ, z) in E. The proof of Theorem 7bis is completed. □

Appendix B : Proofs of Propositions 15 and 16 and Corollary 17.

The proofs of Proposition 15 and 16 are based on the following corollary of item 2 of Proposition 23, which obviously holds under Assumption $(H\pi)$ and Remark 5.

Corollary 25 *Let $\eta \in (-\infty, T]$ and let L be a compact subset of $[a, \infty)$. Then, there exists some $K_{\eta,L} > 0$ such that :*
if for some r, $r' \in [a, \infty)$ and some $t \in [\eta, T]$ the functions w^r and $w^{r'}$ both map $[t, T]$ into L then,

$$\|w^r - w^{r'}\|_{[t,T]} + \|\delta(w^r) - \delta(w^{r'})\|_{[t,T]} \leq K_{\eta,L}|r - r'|.$$

We will also use the following result on the increasing feature of the maximal solution of $BW(r)$ with respect to the terminal condition r.

Proposition 26 *Fix r and r' in $[a, \infty)$ such that $r' < r$. Then, for all $\bar{t} \in [\max\{\tau(r), \tau(r')\}, T)$, we have : $w^{r'} \leq w^r$ on $(\bar{t}, T]$.*

Proof. For ease of notation, we shall write w (resp. w') for w^r (resp. $w^{r'}$). Assume to the contrary that the set $\{t \in (\bar{t}, T] \mid w(t) < w'(t)\}$ is not empty. Then, by continuity of w and w', and since $w(T) = r > r' = w'(T)$, we have

$$\sigma \triangleq \sup\{t \in (\bar{t}, T] \mid w(t) < w'(t)\} \in (\bar{t}, T) \text{ and } w'(\sigma) = w(\sigma).$$

By definition of w and w', we see that, for all $t \in (\bar{t}, \sigma]$,

$$e^{-\lambda t}\dot{\pi}(w(t)) = e^{-\lambda\sigma}\dot{\pi}(w(\sigma)) - \int_t^\sigma e^{-\lambda u}\dot{s}\left(\int_u^T \delta(w(s))ds\right) du$$

and

$$e^{-\lambda t}\dot{\pi}(w'(t)) = e^{-\lambda\sigma}\dot{\pi}(w'(\sigma)) - \int_t^\sigma e^{-\lambda u}\dot{s}\left(\int_u^T \delta(w'(s))ds\right)du.$$

Since $w'(\sigma) = w(\sigma)$, this implies that, for all $t \in (\bar{t}, \sigma)$.,

$$\frac{\dot{\pi}(w(t)) - \dot{\pi}(w'(t))}{\sigma - t} = \frac{e^{\lambda t}}{\sigma - t}\int_t^\sigma e^{-\lambda u}\left[\dot{s}\left(\int_u^T \delta(w'(s))ds\right) - \dot{s}\left(\int_u^T \delta(w(s))ds\right)\right]du.$$

The function

$$u \longmapsto e^{-\lambda u}\left[\dot{s}\left(\int_u^T \delta(w'(s))ds\right) - \dot{s}\left(\int_u^T \delta(w(s))ds\right)\right]$$

being continuous on $(\bar{t}, T]$, we can let t tend to σ in the above equality to obtain

$$\lim_{t\nearrow\sigma}\left\{\frac{[\dot{\pi}(w(t)) - \dot{\pi}(w'(t))]}{\sigma - t}\right\} = \dot{s}\left(\int_\sigma^T \delta(w'(s))ds\right) - \dot{s}\left(\int_\sigma^T \delta(w(s))ds\right). \tag{41}$$

We claim that the right-hand side is negative, so that, for some $\eta > 0$, $\dot{\pi}(w) - \dot{\pi}(w') < 0$, on $(\sigma - \eta, \sigma)$, and hence, by the increasing feature of $\dot{\pi}$ on $[a, \infty)$, $w > w'$ on $(\sigma - \eta, \sigma)$. This will contradict the definition of σ and finally proves Proposition 26.

To see that the above claim holds, recall that δ is increasing on D. Since $\dot{s}$ is increasing, the claim follows from the definition of σ and the fact the continuous functions w and w' satisfy $w(T) = r > r' = w'(T)$. □

Proof of Proposition 15.

Step 1. We first prove that $\tau : r \in [a, \infty) \longmapsto \tau(r) \in [-\infty, T)$ is nondecreasing. Fix r and r' in $[a, \infty)$ such that $r' < r$, and denote w' (resp. w) for $w^{r'}$ (resp. w^r). By Proposition 26, $w' \le w < \infty$ on $(\max\{\tau(r'), \tau(r)\}, T]$. This implies that $\tau(r') \le \max\{\tau(r'), \tau(r)\}$, i.e. $\tau(r') \le \tau(r)$.

Step 2. We now prove that τ is right-continuous as a map from $[a, \infty)$ into $[-\infty, T)$. Assume to the contrary that there exists some $r' \in [a, \infty)$ such that

$$\ell \triangleq \lim_{r\searrow r'} \tau(r) > \tau(r') \in [-\infty, T) \tag{42}$$

For ease of notation, we write w' for $w^{r'}$. Since $\ell > \tau(r')$, $w'(\ell)$ is finite and we can find some real number M such that

$$M > w'(\ell). \tag{43}$$

Since, by Theorem 7, w' is decreasing, we have $M > w'(\ell) > r'$. Combining these inequalities with (42), we deduce that there exists some $r \in [a, M)$ such that

$$0 < r - r' < \frac{M - w'(\ell)}{K_{\ell,[a,M]}}, \tag{44}$$

where $K_{\ell,[a,M]}$ given by Corollary 25. Moreover, since τ is nondecreasing, we have

$$\tau(r) \geq \inf_{z>r'} \tau(z) = \ell > -\infty.$$

It follows that $\tau(r) > -\infty$. We deduce from Theorem 7 that $w^r(\tau(r)+) = \infty$. Now, since w^r is continuous, decreasing, and satisfies $w^r(T) = r < M < \infty$, it follows from the Mean-Value Theorem that there exists some $t_M \in (\tau(r), T)$ such that $w^r(t_M) = M$. Observing that

$$\ell \leq \tau(r) < t_M, \tag{45}$$

it follows from the decreasing feature of w' together with (44) that

$$w^r(t_M) - w'(t_M) > M - w'(\ell) > K_{\ell,[a,M]}(r - r'). \tag{46}$$

We shall now put in light the required contradiction. Since w^r and w' are decreasing, it follows from the equality $w^r(t_M) = M$, (45) and (43), that they both map the interval $[t_M, T]$ into $[a, M]$. Therefore, applying Corollary 25 with $\eta = \ell$ and $L = [a, M]$ we have

$$0 \leq w^r(t_M) - w'(t_M) \leq K_{\ell,[a,M]}(r - r'),$$

which combined with (46) yields the required contradiction.

Step 3. We finally prove that $\lim_{r\to\infty} \tau(r) = T$. Assume to the contrary that

$$\overline{T} \triangleq \lim_{r\to\infty} \tau(r) < T. \tag{47}$$

Considering the sequence $(w^n)_{n\geq a}$, we see that

$$e^{-\lambda\overline{T}}\dot{\pi}(w^n(\overline{T})) - e^{-\lambda T}\dot{\pi}(n) = -\int_{\overline{T}}^{T} e^{-\lambda t}\dot{s}\left(\int_t^T \delta(w^n(s))ds\right)dt.$$

Since, by Theorem 7, w^n is decreasing, we have $w^n(\overline{T}) \geq n$, $\forall n \geq a$. Then letting n tend to ∞ in the previous inequality shows that

$$0 \leq (e^{-\lambda\overline{T}} - e^{-\lambda T})\dot{\pi}(\infty) = \lim_{n\to\infty}\left\{-\int_{\overline{T}}^{T} e^{-\lambda t}\dot{s}\left(\int_t^T \delta(w^n(s))ds\right)dt\right\} \tag{48}$$

where the inequality holds because $\overline{T} < T$ and $\dot{\pi}(\infty) \geq 0$. In order to obtain a contradiction, we shall prove that the right-hand side is negative. Observe that, since $\dot{s}$ is nonnegative,

$$-\int_{\overline{T}}^{T} e^{-\lambda t}\dot{s}\left(\int_t^T \delta(w^n(s))ds\right)dt \leq -\int_{\overline{T}}^{(\overline{T}+T)/2} e^{-\lambda t}\dot{s}\left(\int_t^T \delta(w^n(s))ds\right)dt. \tag{49}$$

Recall that, from Theorem 7, w^n is decreasing and satisfies $w^n(T) = n$. By Remark 5, we deduce that there exists some $n_0 \geq a$ such that for all $n \geq n_0$,

$$\delta(w^n(s)) \geq \delta(w^n(T)) \geq 2\ , \quad \forall s \in [(\overline{T}+T)/2, T].$$

Therefore, for all $n \geq n_0$ and $t \in [\overline{T}, (\overline{T}+T)/2]$

$$\int_t^T \delta(w^n(s))ds \quad \geq \quad \int_{(\overline{T}+T)/2}^T \delta(w^n(s))ds \geq 2\left(T - \frac{\overline{T}+T}{2}\right) \geq T - \overline{T},$$

where we used the nonnegativity of δ. By the increasing feature and nonnegativity of $\dot{s}$, this implies that for all $n \geq n_0$,

$$-\int_{\overline{T}}^{(\overline{T}+T)/2} e^{-\lambda t}\dot{s}\left(\int_t^T \delta(w^n(s))ds\right)dt \leq -\dot{s}(T-\overline{T})\int_{\overline{T}}^{(\overline{T}+T)/2} e^{-\lambda t}dt. \tag{50}$$

From (49) and (50) we deduce that

$$\lim_{n\to\infty}\left\{-\int_{\overline{T}}^T e^{-\lambda t}\dot{s}\left(\int_t^T \delta(w^n(s))ds\right)dt\right\} \leq -\dot{s}(T-\overline{T})\int_{\overline{T}}^{(\overline{T}+T)/2} e^{-\lambda t}dt < 0, \tag{51}$$

where the last inequality holds because $\overline{T} < T$ and $\dot{s}$ is increasing, nonnegative and therefore positive on $(0,\infty)$. The required contradiction follows from (48) and (51).

The second assertion Proposition 15 is a direct consequence of the above steps. □

Proof of Proposition 16.

Assume that $\tau^{-1}(\mathbb{R}^-)$ is not empty.

1. We shall use the following

Remark 27 *It follows from Remark 5 that, for any $r \in [a,\infty)$ the function*

$$t \in (\tau(r), T] \longrightarrow S(t,r) \triangleq \int_t^T \delta(w^r(s))ds \in \mathbb{R}^+$$

is well-defined, decreasing and continuous on $(\tau(r), T]$. In addition, it has has a limit as t goes to $\tau(r)$ which is :

$$\sup_{t\in(\tau(r),T]}\int_t^T \delta(w^r(u))du = \int_{\tau(r)}^T \delta(w^r(u))du \in \mathbb{R}^+ \cup \{\infty\}.$$

Therefore, for any $r \in [a,\infty)$ such that $\tau(r) \leq 0$ we have : $S_0(r) \triangleq \int_0^T \delta(w^r(u))du \in \mathbb{R}^+ \cup \{\infty\}$. The function S_0 is thus well-defined as a map from $\tau^{-1}(\mathbb{R}^-)$ into $\mathbb{R}^+ \cup \{\infty\}$. Besides, from Proposition 26 it follows that it is nondecreasing. Let us prove that it is left-continuous.

Since S_0 is nondecreasing, the limit $\lim_{z\nearrow r} S_0(z)$ exists in $\mathbb{R}^+ \cup \{\infty\}$ for all $r \in \tau^{-1}(\mathbb{R}^-)$. We shall prove that

$$\lim_{z\nearrow r} S_0(z) = S_0(r) \ \text{ in } \mathbb{R}^+ \cup \{\infty\}, \quad \forall r \in \tau^{-1}(\mathbb{R}^-) \cap (a,\infty). \tag{52}$$

Fix $r' \in \tau^{-1}(\mathbb{R}^-) \cap (a, \infty)$.
We first consider the case $S_0(r') \in \mathbb{R}^+$. Fix $\varepsilon > 0$. By 2 of Remark 27, there exists some $t_\varepsilon \in (0, T]$ such that

$$0 \leq S_0(r') - S(t_\varepsilon, r') < \frac{\varepsilon}{3}. \tag{53}$$

Let $r \in \tau^{-1}(\mathbb{R}^-)$ such that $r < r'$. Since by Proposition 15 τ is nondecreasing , w^r and $w^{r'}$ are both defined on $(\tau(r'), T]$. Observe that $w^{r'}$ maps $[t_\varepsilon, T]$ into $[a, w^{r'}(t_\varepsilon)]$. Besides, by Proposition 26, $a \leq w^r \leq w^{r'}$ on $[t_\varepsilon, T]$. Therefore, w^r and $w^{r'}$ map $[t_\varepsilon, T]$ into $[a, w^{r'}(t_\varepsilon)]$. Then, by Corollary 25, for $\eta = \tau(r')$ and $L = [a, w^{r'}(t_\varepsilon)]$, there exists some $K > 0$ such that

$$0 \leq \delta(w^{r'}) - \delta(w^r) \leq K(r' - r) \quad \text{on } [t_\varepsilon, T], \tag{54}$$

where the first inequality follows from Proposition 26 and Remark 5 . Using Proposition 26 and Remark 5 again, we also have

$$0 \leq S_0(r) - S(t_\varepsilon, r) \leq S_0(r') - S(t_\varepsilon, r) < \frac{\varepsilon}{3}. \tag{55}$$

Now, defining

$$\eta(\varepsilon, t_\varepsilon(r')) \triangleq \frac{\varepsilon}{3(T - t_\varepsilon)K},$$

and using (53), (54) and (55), we see that for all $r \in [r' - \eta, r')$

$$\begin{aligned}
|S_0(r') - S_0(r)| &\leq |S_0(r') - S(t_\varepsilon, r')| + |S(t_\varepsilon, r') - S(t_\varepsilon, r)| + |S(t_\varepsilon, r) - S_0(r)| \\
&\leq \frac{2\varepsilon}{3} + \int_{t_\varepsilon}^{T} [\delta(w^{r'}(s)) - \delta(w^r(s))] ds \\
&\leq \frac{2\varepsilon}{3} + (T - t_\varepsilon)K(r' - r) \\
&\leq \varepsilon.
\end{aligned}$$

This proves (52) when $S_0(r') < \infty$.

We now consider the case where $S_0(r') = \infty$. Assume to the contrary that $M \triangleq \lim_{r \nearrow r'} S_0(r) < \infty$. Since S_0 is nondecreasing by (i), we have

$$M = \sup_{r < r'} S_0(r).$$

In order to obtain a contradiction, we shall find some $r \in \tau^{-1}(\mathbb{R}^-)$ such that

$$r < r' \quad \text{and} \quad S_0(r) > M. \tag{56}$$

First notice that since $S_0(r') = \infty$ we have $\tau(r') = 0$. Then, by 2 of Remark 27, there exists some $t_M \in (0, T]$ such that $S(t_M, r') > 2M$. Let $K \triangleq K_{0,[a,w(t_M)]}$ be given by Corollary 25. We set :

$$r \triangleq \max\{r' - \frac{M}{K(T - t_M)}, a\}.$$

By construction, $r < r'$ since by assumption $r' > a$. Then, by the same arguments as in the previous case, we can apply Corollary 25 for $\eta = 0$ and $L = [a, w^{r'}(t_M)]$ on $[t_M, T]$, to obtain

$$\begin{array}{rcl} S(t_M, r) & = & S(t_M, r) - S(t_M, r') + S(t_M, r') \\ & \geq & \displaystyle\int_{t_M}^{T} [\delta(w^r(s)) - \delta(w^{r'}(s))] ds + S(t_M, r') \\ & \geq & (T - t_M) K(r - r') + S(t_M, r') \\ & > & -M + 2M = M. \end{array}$$

This proves (56) and therefore (52) when $S_0(r') = \infty$. The proof of item 1 is completed.

2. From Remark 27, we see that if $\tau(r) < 0$ then $S_0(r) = S(0, r) < \infty$, i.e. $r \in \mathrm{Dom} S_0 \triangleq \{r \in \tau^{-1}(\mathbb{R}^-) \mid S_0(r) < \infty\}$ We shall prove that S_0 is continuous on its domain $\mathrm{Dom} S_0$. Observe that, since S_0 is nondecreasing, $\mathrm{Dom} S_0$ is an interval. Since S_0 is left-continuous, we only have to prove that

$$\text{if } \; r' \in \mathrm{Dom} \; \text{ and } \; r' < \sup(\mathrm{Dom}) \; \text{ then } \; \lim_{r \searrow r'} S_0(r) = S_0(r').$$

Since $r' < \sup(\mathrm{Dom})$, there exists $\bar{r} > r'$ such that $S_0(\bar{r}) < \infty$. Fix $\varepsilon > 0$, then, by Remark 27, there exists some $t_\varepsilon \in (0, T]$ such that

$$0 \leq S_0(\bar{r}) - S(t_\varepsilon, \bar{r}) < \frac{\varepsilon}{3}$$

Now, for all $r \in (r', \bar{r}]$, we have, by Proposition 26 and Remark 5,

$$0 \leq S_0(r') - S(t_\varepsilon, r') \leq S_0(r) - S(t_\varepsilon, r) \leq S_0(\bar{r}) - S(t_\varepsilon, \bar{r}) < \frac{\varepsilon}{3}.$$

Notice that by Proposition 26, for all $r \in [r', \bar{r}]$, w^r maps $[t_\varepsilon, T]$ into $[a, w^{\bar{r}}(t_\varepsilon)]$. The proof can therefore be completed by applying Corollary 25, with $\eta = 0$ and $L = [a, w^{\bar{r}}(t_\varepsilon)]$, to $w^{r'}$ and w^r on $[t_\varepsilon, T]$. This ends the proof of item 2.

3. We now prove that if $\tau(m) > 0$ then, $\sup_{\tau^{-1}(\mathbb{R}^-)} S_0 = \infty$. We claim that

$$\lim_{r \nearrow m} w^r(\tau(m)) = \infty, \tag{57}$$

Then, there exists a sequence (r_n) in $\tau^{-1}(\mathbb{R}^-)$ increasing to m such that

$$\lim_{n \to \infty} w^{r_n}(\tau(m)) = \infty. \tag{58}$$

Using Remark 8, we see that

$$S_0(r_n) = \int_0^T \delta(w^{r_n}(s)) ds \geq \int_0^{\tau(m)} \delta(w^{r_n}(s)) ds \geq \tau(m) \delta(w^{r_n}(\tau(m))).$$

The proof is completed by using (58) and recalling that $\tau(m) > 0$ and that $\lim_{r\to\infty} \delta(r) = \infty$ by Remark 5.

We now prove (57). Observe that, by Proposition 15 and Proposition 26, the function $r \longmapsto w^r(\tau(m))$ is nondecreasing on $[a, m)$. We therefore have

$$M \triangleq \lim_{r \nearrow m} w^r(\tau(m)) = \sup_{r<m} w^r(\tau(m)) \quad \text{and} \quad M \in [a, \infty].$$

We shall prove that $M = \infty$. Assume to the contrary that

$$M = \sup_{r<m} w^r(\tau(m)) < \infty.$$

In order to have a contradiction, we shall find some $r' \in \tau^{-1}(\mathbb{R}^-)$ such that

$$r' < m \quad \text{and} \quad w^{r'}(\tau(m)) > M. \tag{59}$$

First notice that, since $0 > \tau(m) > -\infty$, we have, by Theorem 7, $w^m(\tau(m)+) = \infty$. Therefore, by continuity, there exists some $t_M \in (\tau(m), T]$ such that

$$w^m(t_M) = 2\max\{M, m\}. \tag{60}$$

Let $K \triangleq K_{\tau(m),[a,2\max\{M,m\}]}$ be given by Corollary 25 and define

$$r' \triangleq \max\left\{\left(m - \frac{M}{2K}\right), a\right\}. \tag{61}$$

By definition $r' \in [a, \infty)$. Moreover, since by assumption $\tau(m) > 0$ and $\tau(a) \leq 0$, we have $m > a$. Therefore,

$$r' < m.$$

This is the first requirement of (59). We now prove the second one. Since m is decreasing and $w^m(t_M) = 2\max\{M, m\}$, w^m maps $[t_M, T]$ into $[a, 2\max\{M, m\}]$. Besides, since $r' < m$, it follows by Proposition 26 that $w^{r'} \leq w^m$ on $[t_M, T]$. Hence, $w^{r'}$ also maps $[t_M, T]$ into $[a, 2\max\{M, m\}]$. Therefore, by Corollary 25 with $\eta = \tau(m)$ and $L = [a, 2\max\{M, m\}]$, applied to w^m and $w^{r'}$ we have

$$|w^{r'}(t_M) - w^m(t_M)| \leq K|r' - m|,$$

which reads

$$w^{r'}(t_M) - w^m(t_M) \geq K(r' - m).$$

By (60), we therefore have

$$w^{r'}(t_M) \geq 2\max\{M, m\} + K(r' - m).$$

Noticing that, by (61), $K(r'-m) \geq -M/2$, we obtain

$$w^{r'}(t_M) \geq 2M - M/2 > M.$$

Since $t_M > \tau(m)$ and $w^{r'}$ is decreasing, this implies that

$$w^{r'}(\tau(m)) > M,$$

which ends the proof of (59) and therefore proves (57). The proof of Proposition 16 is completed □

Proof of Corollary 17.

1. Assume that S_0 has a discontinuity at $d \in \tau^{-1}(\mathbb{R}^-)$. Then, since by Proposition 16 S_0 is left-continuous, we see that $d < \sup\{\tau^{-1}(\mathbb{R}^-)\} = m$ and $S_0(d) = \lim_{r \nearrow d} S_0(r)$. Since S_0 is nondecreasing, we must have

$$S_0(d) = \lim_{r \nearrow d} S_0(r) < \lim_{r \searrow d} S_0(r),$$

and therefore $S_0(d) \in \mathbb{R}^+$. It follows that

$$S_0(r) = \infty\ ,\quad \forall r \in \tau^{-1}(\mathbb{R}^-)\ ,\ r > d\ . \tag{62}$$

Otherwise, we would have $[a, r] \subset \mathrm{Dom}(S_0)$ by the increasing feature of S_0. Since, by Proposition 16 S_0 is continuous on its domain, this would implies that S_0 is continuous on $[a, r]$, a contradiction with the definition of d together with the inequality $d < r$. This shows that $\mathrm{Dom}(S_0) = [a, d]$ and that S_0 can not have any other discontinuity point, since it is continuous on its domain. Finally, since by Proposition 16 $[\ \tau(r) < 0 \Rightarrow S_0(r) < \infty\ ]$, it follows from (62) that $\tau(r) = 0$ for all $r \in (d, m]$. By right-continuity of τ, this implies that $\tau(d) = 0$.

2. Assume that $\sup_{\tau^{-1}(\mathbb{R}^-)} S_0 < \infty$. From Proposition 16 we know that if $\tau(m) > 0$ then, $\sup_{\tau^{-1}(\mathbb{R}^-)} S_0 = \infty$. Therefore, $\tau(m) \leq 0$. Now, since $m = \sup \tau^{-1}(\mathbb{R}^-) = \sup\{t \in [a, \infty) \mid \tau(r) \leq 0\}$ and since by Proposition 15 τ is right-continuous, we also have $\tau(m) \geq 0$. This implies that $\tau^{-1}(\mathbb{R}^-) = [a, m]$. The sequel of assertion 2 follows from the increasing feature of S_0. □

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